COMPARISON OF SPACES OF HARDY TYPE FOR THE ORNSTEIN–UHLENBECK OPERATOR

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ABSTRACT. Denote by γ the Gauss measure on $\mathbb{R}^n$ and by $\mathcal{L}$ the Ornstein–Uhlenbeck operator. In this paper we introduce a Hardy space $\mathfrak{h}^1(\gamma)$ of Goldberg type and show that for each u in $\mathbb{R}\setminus\{0\}$ and $r > 0$ the operator $(r\mathcal{I}+\mathcal{L})^{iu}$ is unbounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$. This result is in sharp contrast both with the fact that $(r\mathcal{I}+\mathcal{L})^{iu}$ is bounded from $H^1(\gamma)$ to $L^1(\gamma)$, where $H^1(\gamma)$ denotes the Hardy type space introduced in [MM], and with the fact that in the Euclidean case $(r\mathcal{I}-\Delta)^{iu}$ is bounded from the Goldberg space $\mathfrak{h}^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. We consider also the case of Riemannian manifolds M with Riemannian measure μ. We prove that, under certain geometric assumptions on M, an operator $\mathcal{T}$, bounded on $L^2(\mu)$, and with a kernel satisfying certain analytic assumptions, is bounded from $H^1(\mu)$ to $L^1(\mu)$ if and only if it is bounded from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$. Here $H^1(\mu)$ denotes the Hardy space introduced in [CMM1], and $\mathfrak{h}^1(\mu)$ is defined in Section 4, and is equivalent to a space recently introduced by M. Taylor [T]. The case of translation invariant operators on homogeneous trees is also considered.

1. Introduction

Denote by γ the Gauss measure on $\mathbb{R}^n$, i.e. the probability measure with density $x \mapsto \pi^{-n/2}\,\mathrm{e}^{-|x|^2}$ with respect to the Lebesgue measure.

Harmonic analysis on the measured metric space $(\mathbb{R}^n, d, \gamma)$, where d denotes the Euclidean distance on $\mathbb{R}^n$, has been the object of many investigations. In particular, efforts have been made to study operators related to the Ornstein–Uhlenbeck semigroup, with emphasis on maximal operators [S, GU, MPS1, GMMST2], Riesz transforms [Mu, Gun, M1, M, P, Pe, Gut, GST, FGS, FoS, GMST1, PS, U, DV] and functional calculus [GMST2, GMMST1, MMS].

In [MM] the authors defined an atomic Hardy type space $H^1(\gamma)$ associated to γ. We briefly recall its definition. An Euclidean ball B is called *admissible* if

$$r_B \le \min\big(1, 1/\,|c_B|\big), \tag{1.1}$$

where r_B and c_B denote the radius and the centre of B respectively. An $H^1(\gamma)$-*atom* is either the constant function 1 or a function a in $L^1(\gamma)$, supported in an admissible ball B, such that

$$\|a\|_2 \le \gamma(B)^{-1/2} \qquad \text{and} \qquad \int_{\mathbb{R}^n} a \,\mathrm{d}\gamma = 0, \tag{1.2}$$

where $\|a\|_2$ denotes the norm of a with respect to the Gauss measure. The space $H^1(\gamma)$ is then the vector space of all functions f in $L^1(\gamma)$ that admit a decomposition

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of the form $\sum_j \lambda_j\, a_j$, where the a_j's are $H^1(\gamma)$-atoms and the sequence of complex numbers $\{\lambda_j\}$ is summable. The norm of f in $H^1(\gamma)$ is defined as the infimum of $\sum_j |\lambda_j|$ over all representations of f as above.

Note that $H^1(\gamma)$ is defined much as the atomic space H^1 on spaces of homogeneous type in the sense of R.R. Coifman and G. Weiss [CW], but with a difference. Namely, only the exceptional atom and atoms with "small support", i.e., with support contained in admissible balls, appear in the definition of $H^1(\gamma)$. This difference may appear irrelevant at first sight, but it is, in fact, quite subtle and has important consequences. It is motivated by the fact that the Gauss measure of an open set A in $\mathbb{R}^n$ far away from the origin is concentrated in a "thin" shell near the boundary of A. More precisely, the following quantitative estimate holds [MM, Lemma 3.2 (ii)]: there exist a ball B_0 centred at the origin and a constant C such that for each sufficiently small positive number κ and each open set A contained in B_0^c

$$\gamma\big(\{x\in\mathbb{R}^n : d(x,A^c)\leq \kappa/|x|\}\big)\geq C\,\kappa\,\gamma(A): \tag{1.3}$$

here d denotes the Euclidean distance. Observe that the measured metric space $(\mathbb{R}^n,d,\gamma)$ is nondoubling.

One of the results in [MM] is that if an operator $\mathcal{T}$ is bounded on $L^2(\gamma)$ and has an integral kernel that satisfies a local Hörmander's type integral condition (see (3.1) below), then $\mathcal{T}$ is bounded from $H^1(\gamma)$ to $L^1(\gamma)$, and, consequently on $L^p(\gamma)$ for all p in $(1,2)$. This result applies, for instance, to the imaginary powers of the Ornstein–Uhlenbeck operator (see Section 2 for the precise definition), and, *a fortiori*, to the operators $(\mathcal{I}+\mathcal{L})^{iu}$, where u is in $\mathbb{R}$.

In the Euclidean setting D. Goldberg [G] defined a "local" space of Hardy type $\mathfrak{h}^1(\mathbb{R}^n)$. It is defined much as the atomic Hardy space $H^1(\mathbb{R})$, but atoms are now either standard atoms supported in small balls or square integrable functions supported on large balls satisfying the usual size condition, but without any cancellations.

Recently M. Taylor [T] defined and studied a local Hardy space of Goldberg type in the setting of Riemannian manifolds with bounded geometry. Taylor's definition has a natural analogue in the Gauss setting. In Section 2 we shall define a *local* Hardy space of Goldberg type $\mathfrak{h}^1(\gamma)$ associated to the Gauss measure. The $\mathfrak{h}^1(\gamma)$-atoms are either $H^1(\gamma)$-atoms, or functions a supported in a ball B with $r_B = \min(1,1/|c_B|)$, and satisfying the size condition in (1.2), but *not* the cancellation condition. We shall show that $H^1(\gamma)$ is properly contained in $\mathfrak{h}^1(\gamma)$. Clearly if $\mathcal{T}$ is a bounded linear operator from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$, then it is also bounded from $H^1(\gamma)$ to $L^1(\gamma)$. The converse implication fails. This is one of the main result of this paper.

Specifically, we shall prove that if $\mathcal{T}$ is bounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$ and its kernel $k_{\mathcal{T}}$ satisfies a local Hörmander type condition, then $k_{\mathcal{T}}$ is "uniformly integrable at infinity", i.e.,

$$\sup_{y\in\mathbb{R}^n}\int_{(2B_y)^c}|k(x,y)|\,\mathrm{d}\gamma(x)<\infty;$$

here we denote by B_y the ball with centre y and radius $\min(1,1/|y|)$. As a consequence, in Section 3 we shall prove that if u is in $\mathbb{R}\setminus\{0\}$ and r is positive, then the operator $(r\mathcal{I}+\mathcal{L})^{iu}$, which is bounded from $H^1(\gamma)$ to $L^1(\gamma)$ [MM, Thm 7.2], is unbounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$.

The analysis on the Gauss space described above may be put into a wider perspective. Consider on $\mathbb{R}^n$ the Riemannian distance ρ', whose length element is given by

$$\mathrm{d}s^2 = (1+|x|^2)\,(\mathrm{d}x_1^2+\cdots+\mathrm{d}x_n^2). \tag{1.4}$$

It is not hard to check [CMM2] that balls of radius at most 1 with respect to ρ' are "equivalent" to admissible balls, i.e., balls with respect to the Euclidean distance satisfying condition (1.1). Condition (1.3) is then equivalent to the following

$$\gamma\big(\{x\in\mathbb{R}^n : \rho'(x,A^c)\leq\kappa\}\big)\geq C\,\kappa\,\gamma(A).$$

In the terminology of [CMM2] the measured metric space $(\mathbb{R}^n,\rho',\gamma)$ possesses the so called complementary isoperimetric property (see [CMM2, Section 8]).

A theory of Hardy type spaces on a fairly large class of measured metric spaces (M,ρ,μ) has recently been developed in [CMM1, CMM2]. In these papers we assume that μ is a locally doubling measure, that (M,ρ,μ) possesses an approximate midpoint property (see Section 4 below), and either the isoperimetric or the complementary isoperimetric property, according to whether $\mu(M)$ is infinite or not. When the theory constructed in [CMM2] is applied to the space $(\mathbb{R}^n,\rho',\gamma)$, then the Hardy space $H^1(\mu)$ defined in [CMM2] coincides with the space $H^1(\gamma)$ defined above for the Gauss measure.

Analogues of the local Hardy space of Goldberg type may also be defined in this more general setting. A natural question is whether there are singular integral operators which are bounded from $H^1(\mu)$ to $L^1(\mu)$ but unbounded from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$. In Section 4 we consider the cases where M is either a homogeneous tree, or a Riemannian manifold with spectral gap and Ricci curvature bounded from below.

In the case of trees we prove that an operator invariant with respect to the group of isometries of the tree is bounded from $H^1(\mu)$ to $L^1(\mu)$ if and only if it is bounded from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$.

In the case of manifolds we prove that if $\mathcal{T}$ is a bounded linear operator on $L^2(\mu)$ and has a kernel k satisfying

$$\sup_{y\in M}\int_{B(y,2)^c}|\nabla_1 k(x,y)|\,\mathrm{d}\mu(x)<\infty,$$

where ∇_1 denotes the gradient with respect to the first variable, then $\mathcal{T}$ is bounded from $H^1(\mu)$ to $L^1(\mu)$ if and only if it is bounded from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$.

Furthermore, if M is a *unimodular* Lie group and we endow M with a left invariant Riemannian metric, then a linear operator $\mathcal{T}$ bounded on $L^2(\mu)$ and with kernel satisfying a *local Hörmander type integral condition* (see (4.5) below), is bounded from $H^1(\mu)$ to $L^1(\mu)$ if and only if it is bounded from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$.

We will use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

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2. Notation and background information

The norm of a function f in $L^p(\mu)$ will simply be denoted by $\|f\|_p$. If $\mathcal{T}$ is a bounded linear operator on $L^p(\mu)$, we shall write $|\!|\!|\mathcal{T}|\!|\!|_p$ for its $L^p(\mu)$ operator norm.

We shall consider linear operators $\mathcal{T}$ on various measure spaces (M, μ). When we do, we often associate to $\mathcal{T}$ its kernel, which is defined as follows.

Definition 2.1. Suppose that $\mathcal{T}$ is a bounded linear operator on $L^2(\mu)$ and that k is a function on $M \times M$, locally integrable off the diagonal, such that for every bounded function f with compact support

$$\mathcal{T} f(x) = \int_M k(x,y)\, f(y) \, \mathrm{d}\mu(y) \qquad \forall x \in M \setminus \operatorname{supp} f.$$

Then we say that k is the *kernel* of $\mathcal{T}$ with respect to the measure μ.

In this section and in Section 3 Lebesgue spaces will be with respect to the Gauss measure. In Section 4 we shall also consider Lebesgue spaces with respect to different measures μ on quite general measured metric spaces.

Now we define the Hardy space of Goldberg type $\mathfrak{h}^1(\gamma)$.

Definition 2.2. A *global* atom a is a function in $L^2(\gamma)$ with support contained in a ball B of radius exactly equal to $\min(1, 1/|c_B|)$ such that

$$\|a\|_2 \le \gamma(B)^{-1/2}.$$

A $\mathfrak{h}^1(\gamma)$-atom is either a $H^1(\gamma)$-atom (see the definition at the beginning of the Introduction) or a global atom.

Definition 2.3. The Hardy space of Goldberg type $\mathfrak{h}^1(\gamma)$ is the vector space of all functions f which admit a decompositions of the form

$$f = \sum_j \lambda_j \, a_j, \tag{2.1}$$

where the sequence $\{\lambda_j\}$ is summable and the a_j's are $\mathfrak{h}^1(\gamma)$-atoms. The norm of f in $\mathfrak{h}^1(\gamma)$ is the infimum of $\sum_j |\lambda_j|$ as $\{\lambda_j\}$ varies over all decompositions (2.1) of f.

The $H^1(\gamma)$-atoms and the global atoms that we consider are often referred to as $(1,2)$-atoms. In [MM] it is shown that the space $H^1(\gamma)$ may be defined in terms of the so-called $(1,q)$-atoms, where q is any number in $(1,\infty]$. A similar theory may also be developed for the space $\mathfrak{h}^1(\gamma)$. We omit the details.

In the following proposition we shall make use of the space $BMO(\gamma)$. Recall that an integrable function is in $BMO(\gamma)$ if

$$\|f\|_{BMO(\gamma)} := \|f\|_1 + \sup_B \frac{1}{\gamma(B)} \int_B |f - f_B| \, \mathrm{d}\gamma < \infty,$$

where the supremum is with respect to all admissible balls and

$$f_B = \frac{1}{\gamma(B)} \int_B f \, \mathrm{d}\gamma.$$

It is known that $BMO(\gamma)$ is the Banach dual of $H^1(\gamma)$ [MM, Thm 5.2].

Proposition 2.4. *The inclusion $H^1(\gamma) \subset \mathfrak{h}^1(\gamma)$ is strict.*

Proof. For the sake of simplicity we consider only the case where $n = 1$.

First we show that the monomial $x \mapsto x^2$ is in $BMO(\gamma)$. Indeed, denote by I any admissible interval, with centre c_I and radius r_I. Observe that

$$\begin{aligned} \left|x^2 - c_I^2\right| &= |x - c_I|\; |x + c_I| \\ &\leq r_I\,(r_I + 2\,|c_I|) \qquad \forall x \in I. \end{aligned}$$

If $|c_I| \geq 1$, the right hand side may be estimated by $3r_I\,|c_I|$, which is bounded by 3 because I is admissible. If $|c_I| \leq 1$, then the right hand side is at most $r_I\,(r_I + 2)$, which is dominated by 3, because $r_I \leq 1$. Therefore

$$\int_I \left|x^2 - c_I^2\right| \,\mathrm{d}\gamma(x) \leq 3\,\gamma(I),$$

so that $x \mapsto x^2$ is in $BMO(\gamma)$.

Suppose, by contradiction, that $H^1(\gamma) = \mathfrak{h}^1(\gamma)$. Then, by the closed graph theorem there exists a constant C such that $\|f\|_{H^1(\gamma)} \leq C\|f\|_{\mathfrak{h}^1(\gamma)}$ for all functions f in $\mathfrak{h}^1(\gamma)$. In particular,

$$\|\mathbf{1}_I/\gamma(I)\|_{H^1(\gamma)} \leq C \tag{2.2}$$

for all maximal admissible intervals I.

Since the integral $\int_I x^2 \,\mathrm{d}\gamma$ is absolutely convergent, the pairing between $\mathbf{1}_I$ and the function $x \mapsto x^2$ is given by $\int_I x^2 \,\mathrm{d}\gamma$ (this follows from [MM, Thm 5.2] and the fact that $BMO(\gamma)$ is a lattice, as in [St2, IV.1.2]).

Now observe that, if $|c_I|$ is sufficiently large,

$$\begin{aligned} \|x^2\|_{BMO(\gamma)}\,\|\mathbf{1}_I\|_{H^1(\gamma)} &\geq \int_I x^2 \,\mathrm{d}\gamma(x) \\ &\geq \frac{|c_I|^2}{2}\,\gamma(I), \end{aligned}$$

so that the supremum of the $H^1(\gamma)$-norms of the functions $\mathbf{1}_I/\gamma(I)$ is unbounded as I varies over all maximal admissible intervals, contradicting (2.2). $\square$

3. Imaginary powers of the Ornstein–Uhlenbeck operator

The Ornstein–Uhlenbeck operator $\mathcal{L}$ is the closure in $L^2(\gamma)$ of the operator $\mathcal{L}_0$, defined by

$$\mathcal{L}_0 f = -\frac{1}{2}\,\Delta f + x \cdot \nabla f \qquad \forall f \in C_c^\infty(\mathbb{R}^n),$$

where Δ and ∇ denote the Euclidean Laplacian and gradient respectively. The spectral resolution of the identity of $\mathcal{L}$ is

$$\mathcal{L} f = \sum_{j=0}^{\infty} j\,\mathcal{P}_j f \qquad \forall f \in \mathrm{Dom}(\mathcal{L}),$$

where $\mathcal{P}_j$ is the orthogonal projection onto the linear span of Hermite polynomials of degree j in n variables. For each u in $\mathbb{R}$ consider the sequence $M_u : \mathbb{N} \to \mathbb{C}$, defined by

$$M_u(j) = \begin{cases} 0 & \text{if } j = 0 \\ j^{iu} & \text{if } j = 1, 2, \ldots \end{cases}$$

The family of (spectrally defined) operators $\{M_u(\mathcal{L})\}_{u\in\mathbb{R}}$ will be referred to as *imaginary powers* of the Ornstein–Uhlenbeck operator. They are bounded on $L^p(\gamma)$ for every p in $(1,\infty)$, by the general Littlewood–Paley–Stein theory for generators of symmetric diffusion semigroups [St1]. Sharp estimates of the behavior of their norms on $L^p(\gamma)$ as $|u|$ tends to infinity have been given in [GMMST1] and [MMS], where the estimates are used to prove spectral multiplier theorems. It is also known that they are of weak type $(1,1)$ [GMST2] and bounded from $H^1(\gamma)$ to $L^1(\gamma)$ [MM].

In this section we shall show that for each u in $\mathbb{R}\setminus\{0\}$ the operator $(\mathcal{I}+\mathcal{L})^{iu}$ is unbounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$. Slight modifications of the proof show that a similar result holds for all r in $\mathbb{R}^+$ with $(r\mathcal{I}+\mathcal{L})^{iu}$ in place of $(\mathcal{I}+\mathcal{L})^{iu}$.

This result is in sharp contrast with the Euclidean case. Indeed, it is well known [G] that the operator $(\mathcal{I}-\Delta)^{iu}$ is bounded from $\mathfrak{h}^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

We shall need the following lemma.

Lemma 3.1. *Suppose that $\mathcal{T}$ is a $L^1(\gamma)$-valued linear operator defined on finite linear combinations of $\mathfrak{h}^1(\gamma)$-atoms. The following are equivalent:*

(i) *$\mathcal{T}$ extends to a bounded operator from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$;*
(ii) $\sup\{\|\mathcal{T}a\|_1 : a \text{ is a } \mathfrak{h}^1(\gamma)\text{-atom}\}<\infty$.

Proof. Clearly (i) implies (ii), since every $\mathfrak{h}^1(\gamma)$-atom has $\mathfrak{h}^1(\gamma)$ norm at most 1.

The converse in nontrivial. However, it is not hard to adapt the proof of [MSV, Thm 4.1] to the present case. We omit the details. □

Definition 3.2. Suppose that $\mathcal{T}$ is an operator with kernel k. We say that k satisfies a *local integral condition of Hörmander type* if

$$H_k := \sup_B \sup_{y,y'\in B}\int_{(2B)^c} |k(x,y)-k(x,y')|\,\mathrm{d}\gamma(x)<\infty, \tag{3.1}$$

where $2B$ denotes the ball with the same centre as B and twice the radius and the supremum is taken with respect to all admissible balls.

It is known [MM, Thm 7.2] that if the kernel k satisfies the local Hörmander condition above, then $\mathcal{T}$ extends to a bounded operator from $H^1(\gamma)$ to $L^1(\gamma)$. We aim at showing that $\mathcal{T}$ may be unbounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$. We shall use the following simple criterion.

Proposition 3.3. *Suppose that $\mathcal{T}$ is a bounded linear operator on $L^2(\gamma)$ with kernel k. The following hold:*

(i) *if $\mathcal{T}$ is bounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$ and k satisfies the local Hörmander type condition (3.1), then k satisfies the following estimate*

$$I_\infty := \sup_{y\in\mathbb{R}^n}\int_{(2B_y)^c} |k(x,y)|\,\mathrm{d}\gamma(x)<\infty, \tag{3.2}$$

where, for every y in $\mathbb{R}^n$ we denote by B_y the ball with centre y and radius $\min(1,1/|y|)$*;*

(ii) *if $\mathcal{T}$ is bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and k satisfies (3.2), then $\mathcal{T}$ is bounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$.*

Proof. First we prove (i). Since $\mathcal{T}$ is bounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$, the following holds

$$A := \sup\{\|\mathcal{T}a\|_1 : a \text{ is a global atom}\} < \infty, \tag{3.3}$$

because $\mathfrak{h}^1(\gamma)$-atoms have $\mathfrak{h}^1(\gamma)$-norm at most 1. The function $a_y = \mathbf{1}_{B_y}/\gamma(B_y)$ is a global atom at the scale 1. Notice that

$$\begin{aligned}
\mathcal{T}a_y(x) &= \int_{B_y} k(x,v)\, a_y(v)\,\mathrm{d}\gamma(v) \\
&= \int_{B_y} \big[k(x,v) - k(x,y)\big]\, a_y(v)\,\mathrm{d}\gamma(v) + k(x,y) \int_{B_y} a_y(v)\,\mathrm{d}\gamma(v) \\
&= \frac{1}{\gamma(B_y)} \int_{B_y} \big[k(x,v) - k(x,y)\big]\,\mathrm{d}\gamma(v) + k(x,y).
\end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned}
\int_{(2B_y)^c} |k(x,y)|\,\mathrm{d}\gamma(x) &\leq \sup_{v\in B_y} \int_{(2B_y)^c} \big|k(x,v) - k(x,y)\big|\,\mathrm{d}\gamma(x) + \|\mathcal{T}a_y\|_1 \\
&\leq H_k + \|\mathcal{T}a_y\|_1.
\end{aligned}$$

By taking the supremum over y in $\mathbb{R}^n$, we obtain

$$\sup_{y\in\mathbb{R}^n} \int_{(2B_y)^c} |k(x,y)|\,\mathrm{d}\gamma(x) \leq H_k + A,$$

as required.

Now we prove (ii). Since $\mathcal{T}$ is bounded from $H^1(\gamma)$ to $L^1(\gamma)$, $\mathcal{T}$ is uniformly bounded on $H^1(\gamma)$-atoms. In view of Lemma 3.1 to prove that $\mathcal{T}$ is bounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$ it suffices to show that $\mathcal{T}$ is uniformly bounded on global atoms at the scale 1.

Suppose that a is a global atom at the scale 1, with support contained in B_y. Clearly

$$\|\mathcal{T}a\|_1 = \|\mathbf{1}_{4B_y}\mathcal{T}a\|_1 + \|\mathbf{1}_{(4B_y)^c}\mathcal{T}a\|_1. \tag{3.5}$$

It is not hard to check that there exists a constant C, independent of y in $\mathbb{R}^n$, such that $\gamma(4B_y)^{1/2} \leq C\,\gamma(B_y)^{1/2}$ (see [MM, prop. 2.1 (ii)]). Therefore

$$\begin{aligned}
\|\mathbf{1}_{4B_y}\,\mathcal{T}a\|_1 &\leq \gamma(4B_y)^{1/2}\ \|\mathcal{T}a\|_2 \\
&\leq C\,\gamma(B_y)^{1/2}\ |\!|\!|\mathcal{T}|\!|\!|_2\,\|a\|_2 \\
&\leq C\,|\!|\!|\mathcal{T}|\!|\!|_2.
\end{aligned}$$

Furthermore, since $v \in B_y$ implies that $2B_v \subset 4B_y$,

$$\begin{aligned}
\|\mathbf{1}_{(4B_y)^c}\mathcal{T}a\|_1 &\leq \int_{\mathbb{R}^n} \mathrm{d}\gamma(v)\ |a(v)| \int_{(2B_v)^c} |k(x,y)|\,\mathrm{d}\gamma(x) \\
&\leq I_\infty\,\|a\|_1 \\
&\leq I_\infty.
\end{aligned}$$

Hence

$$\|\mathcal{T}a\|_1 \leq C\,|\!|\!|\mathcal{T}|\!|\!|_2 + I_\infty,$$

with C independent of a, as required to conclude the proof of the proposition. □

Fix u in $\mathbb{R}\setminus\{0\}$ and r in $\mathbb{R}^+$. The kernel k of the operator $(\mathcal{L}+r\mathcal{I})^{iu}$ (with respect to the Gauss measure) is given by

$$k(x,y)=\frac{1}{\Gamma(iu)}\int_0^\infty t^{-iu-1}\,\mathrm{e}^{-rt}\,h_t(x,y)\,\mathrm{d}t \qquad \forall x,y\in\mathbb{R}^n,\ x\neq y, \tag{3.6}$$

where Γ denotes the Euler function and h_t is the Mehler's kernel, i.e. the kernel of the operator $\exp(-t\mathcal{L})$ [GMMST1] with respect to the Gauss measure. Recall the formula

$$h_t(x,y)=\frac{1}{(1-\mathrm{e}^{-2t})^{n/2}}\ \exp\Big[|y|^2-\frac{|\mathrm{e}^{-t}\,x-y|^2}{1-\mathrm{e}^{-2t}}\Big] \tag{3.7}$$

where t is in $\mathbb{R}^+$ and x and y are in $\mathbb{R}^n$. We perform the change of variables $t=\log((1+s)/(1-s))$ in (3.6). This change of variables, which was first introduced in [GMMST1], transforms the Mehler kernel to

$$\widetilde{h}_s(x,y)=\frac{(1+s)^n}{(4s)^{n/2}}\ \exp\Big[\frac{|x|^2+|y|^2}{2}-\frac{1}{4}\,(s\,|x+y|^2+s^{-1}\ |x-y|^2)\Big], \tag{3.8}$$

and for $r=1$ the kernel k is expressed via the following formula

$$k(x,y)=\frac{1}{\Gamma(iu)}\int_0^1 \frac{g_u(s)}{1+s}\,\mathrm{e}^{-Q_s(x,y)}\,\frac{\mathrm{d}s}{s^{1/2}} \qquad \forall x,y\in\mathbb{R}^n,\ x\neq y. \tag{3.9}$$

where Q_s denotes the quadratic form

$$Q_s(x,y)=\frac{1}{2}\,(|x|^2+|y|^2)-\frac{1}{4}\,\Big(\frac{|x-y|^2}{s}+s\,|x+y|^2\Big),$$

and $g_u:(0,1)\to\mathbb{C}$ is the function defined by

$$g_u(s)=\Big[\log\Big(\frac{1+s}{1-s}\Big)\Big]^{-iu-1}.$$

Following [GMMST1], for every a in $\mathbb{R}^+$ define the function F_a

$$F_a(s)=-a(s-1)^2/4s \qquad \forall s\in\mathbb{R}^+$$

and

$$I(a,\sigma)=\int_0^1 g_u(s)\,\frac{\mathrm{e}^{F_a(s/\sigma)}}{1+s}\,\frac{\mathrm{d}s}{s^{1/2}}.$$

It is straightforward, though tedious, to check that in the case where $n=1$ the following formula holds

$$k(x,y)=\mathrm{e}^{x^2}\,I\big(\big|x^2-y^2\big|\,,|x-y|\,/\,|x+y|\big). \tag{3.10}$$

The following lemma, which is reminiscent of [GMMST1, Lemma 4.2], will be the key to obtain precise estimates of k.

Lemma 3.4. *There exist constants $C>0$ and $a_0\geq 1$ such that*

$$|I(a,\sigma)|\geq\frac{C}{\sqrt{a\,\sigma}} \qquad \forall a\in[a_0,\infty)\quad\forall\sigma\in(0,1/2].$$

Proof. It will be convenient to define two more functions, J and H, by the formulae

$$J(a,\sigma)=g_u(\sigma)\int_{\sigma/2}^{2/3}\frac{\mathrm{e}^{F_a(s/\sigma)}}{1+s}\,\frac{\mathrm{d}s}{s^{1/2}} \qquad\text{and}\qquad H(a,\sigma)=I(a,\sigma)-J(a,\sigma).$$

We claim that there exist positive constants C and $M > 0$ such that

$$|J(a,\sigma)| \geq \frac{C}{\sqrt{a\sigma}} \qquad \text{and} \qquad |H(a,\sigma)| \leq \frac{M}{a\sqrt{\sigma}} \qquad \forall a \in [1,\infty) \quad \forall \sigma \in (0,1/2].$$

The required estimate on I will follow directly from the claim.

To prove the claim, define H^1, H^2 and H^3 by

$$H^1(a,\sigma) = \int_0^{\sigma/2} g_u(s)\, \frac{\mathrm{e}^{F_a(s/\sigma)}}{1+s}\, \frac{\mathrm{d}s}{s^{1/2}} \qquad\qquad H^2(a,\sigma) = \int_{2/3}^{1} g_u(s)\, \frac{\mathrm{e}^{F_a(s/\sigma)}}{1+s}\, \frac{\mathrm{d}s}{s^{1/2}}$$

and

$$H^3(a,\sigma) = \int_{\sigma/2}^{2/3} \big(g_u(s) - g_u(\sigma)\big)\, \frac{\mathrm{e}^{F_a(s/\sigma)}}{1+s}\, \frac{\mathrm{d}s}{s^{1/2}}.$$

Clearly $H = H^1 + H^2 + H^3$. Observe that there is a constant C such that

$$\Big[\log\Big(\frac{1+s}{1-s}\Big)\Big]^{-1} (1+s)^{-1} \leq C\, s^{-1} \qquad \forall s \in (0,1/4].$$

Hence there exist positive constants C and c such that

$$\begin{aligned}
\big|H^1(a,\sigma)\big| &\leq \int_0^{\sigma/2} \Big[\log\Big(\frac{1+s}{1-s}\Big)\Big]^{-1} \frac{s^{-1/2}}{1+s}\, \mathrm{e}^{F_a(s/\sigma)}\, \mathrm{d}s \\
&\leq C \int_0^{\sigma/2} s^{-3/2}\, \mathrm{e}^{F_a(s/\sigma)}\, \mathrm{d}s \\
&\leq C \int_0^{\sigma/2} s^{-3/2}\, \mathrm{e}^{-ca\sigma/s}\, \mathrm{d}s \\
&\leq C\, (a\sigma)^{-1/2} \int_{2ca}^{\infty} s^{-1/2}\, \mathrm{e}^{-s}\, \mathrm{d}s \\
&\leq C\, a^{-1}\, \sigma^{-1/2}\, \mathrm{e}^{-2ca} \qquad \forall a \in [1,\infty) \quad \forall \sigma \in (0,1/2].
\end{aligned} \tag{3.11}$$

A similar computation shows that there exists $C > 0$ such that

$$\big|H^2(a,\sigma)\big| \leq C\, a^{-1}\, \sigma^{-1/2} \qquad \forall a \in [1,\infty) \quad \forall \sigma \in (0,1/2]. \tag{3.12}$$

Now we estimate $H^3(a,\sigma)$. Note that there exists $C > 0$ such that

$$\Big|\frac{\mathrm{d}}{\mathrm{d}s} g_u(s)\Big| \leq C_u \Big[\log\Big(\frac{1+\sigma/2}{1-\sigma/2}\Big)\Big]^{-2} \qquad \forall s \in (\sigma/2, 2/3). \tag{3.13}$$

Hence, by the mean value theorem, we have that

$$\begin{aligned}
\big|H^3(a,\sigma)\big| &\leq C_u \int_{\sigma/2}^{2/3} |s-\sigma|\, \Big[\log\Big(\frac{1+\sigma/2}{1-\sigma/2}\Big)\Big]^{-2} \frac{s^{-1/2}}{1+s}\, \mathrm{e}^{F_a(s/\sigma)}\, \mathrm{d}s \\
&\leq C\, \sigma^{-2} \int_{\sigma/2}^{2/3} |s-\sigma|\; s^{-1/2}\, \mathrm{e}^{F_a(s/\sigma)}\, \mathrm{d}s \\
&= C\, \sigma^{-1/2} \int_{1/2}^{2/(3\sigma)} \frac{|v-1|}{\sqrt{v}}\, \mathrm{e}^{F_a(v)}\, \mathrm{d}v.
\end{aligned} \tag{3.14}$$

Now, observe that there exist positive constants c and C such that

$$c\, a\, (v-1)^2 \leq F_a(v) \leq C\, a\, (v-1)^2 \qquad \forall v \in [1/2, 4/3] \quad \forall a \in [1,\infty), \tag{3.15}$$

and

$$c\, a\, v \leq F_a(v) \leq C\, a\, v \qquad \forall v \in [4/3, \infty) \quad \forall a \in [1,\infty). \tag{3.16}$$

We split the last integral above as the sum of the integrals over the intervals $[1/2, 4/3]$ and $[4/3, 2/(3\sigma)]$. By (3.15) the first of these two integrals is dominated by

$$\begin{aligned} C \int_{1/2}^{4/3} |v-1| \, \mathrm{e}^{ca(v-1)^2)} \, \mathrm{d}v &\leq C \int_{-\infty}^{\infty} |v-1| \, \mathrm{e}^{-ca(v-1)^2} \, \mathrm{d}v \\ &= C\, a^{-1} \qquad \forall a \in [1, \infty), \end{aligned}$$

Similarly, by (3.16), the second integral may be estimated by

$$\begin{aligned} C \int_{4/3}^{\infty} \sqrt{v}\, \mathrm{e}^{-cav} \, \mathrm{d}v &\leq C\, a^{-3/2} \int_{4a/3}^{\infty} \sqrt{u}\, \mathrm{e}^{-cu} \, \mathrm{d}u \\ &\leq C\, a^{-1}\, \mathrm{e}^{-4ca/3} \qquad \forall a \in [1, \infty), \end{aligned}$$

The last two estimates, combined with (3.14), imply

$$\left| H^3(a, \sigma) \right| \leq C\, a\, \sigma^{-1/2} \qquad \forall a \in [1, \infty) \quad \forall \sigma \in (0, 1/2].$$

By combining this estimate with (3.11), (3.12), we get the desired bound for $H(a, \sigma)$.

Now we estimate $J(a, \sigma)$. Observe that

$$\begin{aligned} |J(a, \sigma)| &= |g_u(\sigma)| \int_{\sigma/2}^{2/3} \frac{s^{-1/2}}{1+s} \, \mathrm{e}^{F_a(s/\sigma)} \, \mathrm{d}s \\ &\geq C\, |g_u(\sigma)| \int_{\sigma/2}^{2/3} \mathrm{e}^{F_a(s/\sigma)} \, \mathrm{d}s \\ &\geq C \left[\log\Bigl(\frac{1+\sigma}{1-\sigma} \Bigr) \right]^{-1} \sqrt{\frac{\sigma}{a}} \\ &\geq C\, (a\, \sigma)^{-1/2} \qquad \forall a \in [1, \infty) \quad \forall \sigma \in (0, 1/2], \end{aligned}$$

as required. $\square$

Theorem 3.5. *For each u in $\mathbb{R} \setminus \{0\}$ and for each r in $\mathbb{R}^+$ the operator $(r\mathcal{I} + \mathcal{L})^{iu}$ is unbounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$.*

Proof. We prove the result when $r = 1$. The modifications needed to prove the result for $r > 0$ are straighforward and omitted.

A slight modification of the proof of [MM, Thm 7.2] shows that the kernel k of $(\mathcal{I} + \mathcal{L})^{iu}$ satisfies Hörmander's type condition (3.1). Thus, by Proposition 3.3, to prove the theorem it suffices to show that

$$\lim_{|y| \to \infty} \int_{(2B_y)^c} |k(x, y)| \, \mathrm{d}\gamma(x) = \infty, \tag{3.17}$$

where B_y denotes the ball with centre y and radius $\min(1, 1/|y|)$.

We shall give the details only in the case where $n = 1$. The proof in the case where $n \geq 2$ is more technical, but it follows the same lines. See also the proof of [GMMST1, Proposition 4.4], where similar computations are made in all dimensions and the differences between the one dimensional and the higher dimensional cases are explained in detail.

By (3.10), it suffices to prove that the function

$$y \mapsto \int_{(2B_y)^c} \left| I\bigl(\left| x^2 - y^2 \right|, |x-y| \,/\, |x+y| \bigr) \right| \mathrm{d}x$$

is unbounded. We may restrict the domain of integration to the set where y is large and positive, and x is in the interval $(y-1, y-a_0/y)$, with $a_0 \geq 2$ (a_0 is as in the statement of Lemma 3.4). Then we must prove that

$$\lim_{y\to\infty} \int_{y-1}^{y-a_0/y} \big|I\big(y^2-x^2, (y-x)/(x+y)\big)\big| \,\mathrm{d}x = \infty. \tag{3.18}$$

Note that in the interval $(y-1, y-a_0/y)$

$$\big|I\big(y^2-x^2, (y-x)/(x+y)\big)\big| \geq C\,(y-x)^{-1}.$$

Indeed, in that interval $y^2-x^2 \geq (a_0/y)\,(x+y) \geq a_0$, and $(y-x)/(x+y) \leq 1/2$, so that Lemma 3.4 may be applied, and the estimate above follows. Therefore the limit in (3.18) is estimated from below by

$$\lim_{y\to\infty} \int_{y-1}^{y-a_0/y} (y-x)^{-1}\,\mathrm{d}x = \infty,$$

as required to conclude the proof of the theorem. Now (3.18) follows directly from this estimate. $\square$

Remark 3.6. If r is in $\mathbb{R}^+$, then the operator $(r\mathcal{I}+\mathcal{L})^{iu}$ is bounded from $H^1(\gamma)$ to $L^1(\gamma)$. The proof of this fact may be obtained by a straightforward adaptation of the proof of [MM, Thm 7.2]. Since $(r\mathcal{I}+\mathcal{L})^{iu}$ is unbounded from $\mathfrak{h}^1(\gamma)$ to $L^1(\gamma)$ by Theorem 3.5, the inclusion $H^1(\gamma) \subset \mathfrak{h}^1(\gamma)$ is strict, thereby giving another proof of Proposition 2.4.

4. Measured metric spaces

We recall briefly the relevant definition and refer to [CMM1, CMM2] and the references therein for every unexplained notation and terminology and for more on measured metric spaces.

Suppose that (M,ρ,μ) is a measured metric space. In particular, we assume that (M,ρ) is a metric space, that μ is a regular Borel measure on M with the property that $\mu(M) > 0$ and every ball has finite measure. We assume throughout that M is *unbounded*. We denote by $\mathcal{B}$ the family of all balls on M. For each B in $\mathcal{B}$ we denote by c_B and r_B the centre and the radius of B respectively, and by κB the ball with centre c_B and radius $\kappa\, r_B$. For each b in $\mathbb{R}^+$, we denote by $\mathcal{B}_b$ the family of all balls B in $\mathcal{B}$ such that $r_B \leq b$. For any subset A of M and each κ in $\mathbb{R}^+$ we denote by A_κ and A^κ the sets

$$\big\{x \in A : \rho(x, A^c) \leq \kappa\big\} \qquad \text{and} \qquad \big\{x \in A : \rho(x, A^c) > \kappa\big\}$$

respectively.

We say that the measured metric space (M,ρ,μ) possesses the *local doubling property* (LDP) if for every b in $\mathbb{R}^+$ there exists a constant D_b such that

$$\mu\big(2B\big) \leq D_b\, \mu\big(B\big) \qquad \forall B \in \mathcal{B}_b. \tag{4.1}$$

We say that the measured metric space (M,ρ,μ) with $\mu(M) = \infty$ possesses the *isoperimetric property* (I) if there exist κ_0 and C in $\mathbb{R}^+$ such that for every bounded open set A

$$\mu\big(A_\kappa\big) \geq C\,\kappa\,\mu(A) \qquad \forall \kappa \in (0, \kappa_0]. \tag{4.2}$$

It is known [CMM1] that if M is a complete Riemannian manifold, the isoperimetric property (defined in terms of the Riemannian distance and the Riemannian volume) is equivalent to the positivity of Cheeger's isoperimetric costant $h(M)$, defined by

$$h(M) = \inf \frac{\sigma\big(\partial(A)\big)}{\mu(A)},$$

where the infimum runs over all bounded open sets A with smooth boundary. Here σ denotes the induced Riemannian measure on ∂A. Moreover, if the Ricci curvature of M is bounded from below, both properties are equivalent to the existence of a spectral gap for the Laplacian.

The analogue of the isoperimetric property for measured metric spaces of finite measure is the so-called complementary isoperimetric inequality, which we now define. We say that a measured metric space (M, ρ, μ) of finite measure possesses the *complementary isoperimetric property* (I^c) if there exist a ball B_0 in M, κ_0 and C in $\mathbb{R}^+$ such that for every bounded open set A contained in $M \setminus \bar{B}_0$

$$\mu\big(A_\kappa\big) \geq C\, \kappa\, \mu(A) \qquad \forall \kappa \in (0, \kappa_0]. \tag{4.3}$$

We say that the measured metric space (M, ρ, μ) possesses the *property* (AMP) (approximate midpoint property) if there exist R_0 in $[0, \infty)$ and β in $(1/2, 1)$ such that for every pair of points x and y in M with $\rho(x, y) > R_0$ there exists a point z in M such that $\rho(x, z) < \beta\, \rho(x, y)$ and $\rho(y, z) < \beta\, \rho(x, y)$.

Clearly every length measured metric space possesses property (AMP). The measured metric space $(\mathbb{R}^n, \rho', \gamma)$ (ρ' is as in (1.4)) is a locally doubling measured metric space with the complementary isoperimetric and the approximate midpoint property.

We briefly recall the definition of the Hardy space $H^1(\mu)$ in this setting [CMM1, CMM2].

Definition 4.1. A (standard) *atom* a is a function in $L^2(\mu)$ supported in a ball B in $\mathcal{B}$ such that

$$\|a\|_2 \leq \mu(B)^{-1/2} \qquad \text{and} \qquad \int_B a \,\mathrm{d}\mu = 0.$$

Definition 4.2. Suppose that $\mu(M) = \infty$. The *Hardy space* $H^1(\mu)$ is the space of all functions g in $L^1(\mu)$ that admit a decomposition of the form

$$g = \sum_{k=1}^{\infty} \lambda_k\, a_k,$$

where a_k is an atom *supported in a ball B of radius at most* 1, and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. The norm $\|g\|_{H^1(\mu)}$ of g is the infimum of $\sum_{k=1}^{\infty} |\lambda_k|$ over all decompositions of g as above.

In the case where μ is finite in addition to the standard atoms defined above there is also an *exceptional atom*, i.e. the constant function $1/\mu(M)$. The *Hardy space* $H^1(\mu)$ is defined as in the case where $\mu(M) = \infty$, but now atoms are either standard atoms or the exceptional atom. These atoms will be referred to as $H^1(\mu)$-atoms.

To avoid technicalities we assume throughout that $R_0/(1-\beta) < 1$. In view of [CMM1, Prop. 4.3] and [CMM2, Prop. 3.4 (i)] this ensures that the Hardy space $H^1(\mu)$ defined above is scale invariant in the following sense. For $b > (R_0/(1-\beta)$

we may consider an Hardy space $H^1_b(\mu)$ defined as in Definition 4.2, but where atoms are supported in balls of radius at most b instead that 1. With this notation the space $H^1(\mu)$ defined above would be denoted by $H^1_1(\mu)$. It is a nontrivial fact that the spaces $H^1_1(\mu)$ and $H^1_b(\mu)$ agree as vector spaces and that their norms are equivalent.

Now we define the Goldberg type space $\mathfrak{h}^1(\mu)$ in this setting.

Definition 4.3. A *global* atom a (at the scale 1) is a function in $L^2(\mu)$ with support contained in a ball B of radius exactly equal to 1 such that

$$\|a\|_2 \leq \mu(B)^{-1/2}.$$

An $\mathfrak{h}^1(\mu)$-atom is either an $H^1(\mu)$-atom or a global atom.

Definition 4.4. The Hardy space of Goldberg type $\mathfrak{h}^1(\mu)$ is the vector space of all functions f which admits a decompositions of the form

$$f = \sum_j \lambda_j \, a_j, \tag{4.4}$$

where the sequence $\{\lambda_j\}$ is summable and the a_j's are $\mathfrak{h}^1(\mu)$-atoms. The norm of f in $\mathfrak{h}^1(\mu)$ is the infimum of $\sum_j |\lambda_j|$ as $\{\lambda_j\}$ varies over all decompositions (4.4) of f.

If μ is infinite, then $H^1(\mu)$ is contained in the space of integrable functions with integral 0. Since global atoms in $\mathfrak{h}^1(\mu)$ are integrable functions with possibly nonzero integral, the strict inclusion $H^1(\mu) \subset \mathfrak{h}^1(\mu)$ holds also in this case.

An equivalent space has been defined and studied on Riemannian manifolds with bounded geometry by M. Taylor [T]. In fact, the definition of Taylor is different from that adopted above (see [T, Section 2]), but it is straightforward to check that the two definitions are equivalent, i.e., the corresponding spaces agree, with equivalent norms.

Assume that $\mathcal{T}$ is a bounded linear operator on $L^2(\mu)$ with kernel k (see Section 2 for the definition). In [CMM1, Thm 8.2] it has been proved that, if k satisfies the following *local Hörmander type condition*

$$H_k = \sup_B \sup_{y,y' \in B} \int_{(2B)^c} |k(x,y) - k(x,y')| \, \mathrm{d}\mu(x) < \infty, \tag{4.5}$$

where the supremum is taken over all balls B of radius at most 1, then $\mathcal{T}$ extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$. It is natural to speculate under what conditions the operator $\mathcal{T}$ extends to a bounded operator from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$. The following is the analogue of Proposition 3.3 above.

Proposition 4.5. *Suppose that $\mathcal{T}$ is a bounded linear operator on $L^2(\mu)$ with kernel k. The following hold:*

(i) *if $\mathcal{T}$ is bounded from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$ and k satisfies the local Hörmander type condition (4.5), then k satisfies the following estimate*

$$I_\infty := \sup_{y \in M} \int_{B(y,2)^c} |k(x,y)| \, \mathrm{d}\mu(x) < \infty; \tag{4.6}$$

(ii) *if $\mathcal{T}$ is bounded from $H^1(\mu)$ to $L^1(\mu)$ and k satisfies (4.6), then $\mathcal{T}$ is bounded from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$.*

Proof. The proof is, *mutatis mutandis*, the same as the proof of Proposition 3.3. We only need to replace the ball B_y in that proof with the ball $B(y,1)$. We omit the details. $\square$

4.1. **Homogeneous trees.** We now show that there are cases in which boundedness from $H^1(\mu)$ to $L^1(\mu)$ is equivalent to boundedness from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$. This is in sharp contrast with the case of the Gauss measure which has been analysed in the Section 3.

Denote by $\mathfrak{X}$ a homogeneous tree, i.e., a graph, with no loops, in which every vertex x has the same number, $q+1$ say, of adjacent vertices, called nearest neighbours of x. When x and y are adjacent vertices, we shall write $x \sim y$. Denote by μ the *counting measure* on $\mathfrak{X}$, and by ρ one-half of the natural distance on $\mathfrak{X}$. Thus, two adjacent vertices have distance $1/2$. The reason for this apparently unnatural definition of distance is that if the distance of two adjacent vertices were equal to 1, then the only atom supported on any ball of radius at most 1 would be the trivial atom. We could, of course, consider balls of radius at most 2, but then this would require new definitions and there would not be uniformity with the Gaussian case and the case of manifolds.

Denote by G the group of isometries of $\mathfrak{X}$ (see [FTN] for information on G) and fix a reference point o in $\mathfrak{X}$. We shall consider only G-invariant linear operators acting on function spaces on $\mathfrak{X}$. If $\mathcal{T}$ is such an operator, then its kernel k satisfies the following

$$k(x,y) = k(g\cdot x, g\cdot y) \qquad \forall g \in G \quad \forall x,y \in \mathfrak{X},$$

so that $k(x,y)$ depends, in fact, only on $\rho(x,y)$. As a consequence, the local Hörmander type condition (4.5) may be reformulated thus

$$\max_{y\sim o} \sum_{x:\rho(x,o)\geq 2} |k(x,y)-k(x,o)| < \infty. \tag{4.7}$$

Proposition 4.6. *Suppose that $\mathcal{T}$ is a G-invariant linear operator defined on functions on $\mathfrak{X}$ with finite support and denote by k its kernel. The following hold:*

(i) *if $\mathcal{T}$ extends to a bounded operator from $H^1(\mathfrak{X})$ to $L^1(\mathfrak{X})$, then k satisfies the Hörmander type integral condition (4.7);*
(ii) *if k satisfies the local Hörmander type integral condition (4.7), then*

$$\sum_{x\in\mathfrak{X}} |k(x,o)| < \infty.$$

Hence $\mathcal{T}$ is bounded on $L^1(\mathfrak{X})$;
(iii) *$\mathcal{T}$ extends to a bounded operator from $H^1(\mathfrak{X})$ to $L^1(\mathfrak{X})$ if and only if $\mathcal{T}$ extends to a bounded operator from $\mathfrak{h}^1(\mathfrak{X})$ to $L^1(\mathfrak{X})$.*

Proof. First we prove (i). For each pair y, y_0 of adjacent vertices, define the function a_{y,y_0} by $\delta_y - \delta_{y_0}$. Clearly a_{y,y_0} is a multiple of an $H^1(\mu)$-atom, and

$$\begin{aligned}\mathcal{T} a_{y,y_0}(w) &= \sum_{z\in\mathfrak{X}} a_{y,y_0}(z)\, k(w,z) \\ &= k(w,y) - k(w,y_0) \qquad \forall w \in \mathfrak{X}.\end{aligned}$$

The assumption $\mathcal{T}$ bounded from $H^1(\mu)$ to $L^1(\mu)$ forces $\|\mathcal{T}a_{y,y_0}\|_1$ to be uniformly bounded with respect to all y and y_0 such that $y \sim y_0$. Thus,

$$\max_{y\sim y_0} \sum_{w\in\mathfrak{X}} |k(w,y)-k(w,y_0)| \leq C \,|\!|\!|\mathcal{T}|\!|\!|_{H^1;L^1},$$

as required.

Next we prove (ii). Fix a reference point o, and denote by $\eta : \mathfrak{X} \to \mathbb{C}$ the function defined by

$$\eta(x) = k(x,o).$$

Clearly η is a radial function, i.e., it depends only on the distance of x from o. Suppose that x is a point at distance j from o and that $y \sim o$. Then the distance from x to y is either $j-1/2$ or $j+1/2$. Furthermore, there are exactly q vertices y adjacent to o such that $\rho(x,y) = j+1/2$ and only one vertex adjacent to o such that $\rho(x,y) = j-1/2$. Now, by summing in polar co-ordinates centred at o, we see that

$$\begin{aligned}\sum_{y\sim o} \sum_{x:\rho(x,o)\geq 2} |k(x,y)-k(x,o)| &= \sum_{2j=4}^{\infty} \sum_{x:\rho(x,o)=j} \sum_{y\sim o} |k(x,y)-k(x,o)| \\ &= \sum_{2j=4}^{\infty} \sum_{x:\rho(x,o)=j} \sum_{x'\sim x} |\eta(x')-\eta(x)| .\end{aligned} \tag{4.8}$$

Observe that

$$\sum_{x'\sim x} |\eta(x')-\eta(x)| \geq \Big[\sum_{x'\sim x} |\eta(x')-\eta(x)|^2\Big]^{1/2}.$$

and recall that the right hand side is just $|\nabla\eta(x)|$, by definition of length of the gradient of η (see, for instance, [CG, p. 658]). Then, by summing both sides with respect to all x such that $\rho(x,o) \geq 2$ and using (4.8), we obtain

$$\sum_{y\sim o} \sum_{x:\rho(x,o)\geq 2} |k(x,y)-k(x,o)| \geq \sum_{x:\rho(x,o)\geq 2} |\nabla\eta(x)| .$$

Clearly

$$\sum_{x:\rho(x,o)<2} |\nabla\eta(x)| < \infty,$$

because the sum is finite, so that we may conclude that $\|\,|\nabla\eta|\,\|_1$ is finite. By the isoperimetric property [Ch1, Thm VI.4.2], $\|\,|\nabla\eta|\,\|_1 \geq C\,\|\eta\|_1$, hence η is in $L^1(\mathfrak{X})$, i.e.,

$$\sum_{x\in\mathfrak{X}} |k(x,o)| < \infty,$$

as required. This condition clearly implies that $\mathcal{T}$ is bounded on $L^1(\mathfrak{X})$, and the proof of (ii) is complete.

Finally, to prove (iii), observe that if $\mathcal{T}$ extends to a bounded operator from $\mathfrak{h}^1(\mathfrak{X})$ to $L^1(\mathfrak{X})$, then clearly $\mathcal{T}$ extends to a bounded operator from $H^1(\mathfrak{X})$ to $L^1(\mathfrak{X})$.

Conversely, suppose that $\mathcal{T}$ extends to a bounded operator from $H^1(\mathfrak{X})$ to $L^1(\mathfrak{X})$. By (i) its kernel k satisfies the local Hörmander integral condition. By (ii) the operator $\mathcal{T}$ extends to a bounded operator on $L^1(\mathfrak{X})$, hence, *a fortiori*, from $\mathfrak{h}^1(\mathfrak{X})$ to $L^1(\mathfrak{X})$. □

4.2. **Riemannian manifolds.** Finally, we consider a connected noncompact Riemannian manifold M, with spectral gap and Ricci curvature bounded from below. Recall that a Riemannian manifold M is said to have spectral gap if the bottom of the L^2 spectrum of the associated Laplace–Beltrami operator is strictly positive. Such manifolds possess the isoperimetric property (see, for instance, [CMM1, Section 9] and the references therein).

Denote by μ the Riemannian measure of M.

Theorem 4.7. *Suppose that M is as above, that $\mathcal{T}$ is a bounded linear operator on $L^2(\mu)$ and that its kernel k satisfies*

$$C_0 := \sup_{y \in M} \int_{B(y,2)^c} |\nabla_1 k(x,y)| \, \mathrm{d}\mu(x) < \infty. \tag{4.9}$$

Then $\mathcal{T}$ extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$ if and only if $\mathcal{T}$ extends to a bounded operator from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$.

Proof. Clearly if $\mathcal{T}$ extends to a bounded operator from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$, then it extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$.

Conversely, suppose that $\mathcal{T}$ extends to a bounded operator from $H^1(\mu)$ to $L^1(\mu)$. Then it is uniformly bounded on $H^1(\mu)$-atoms. Hence to conclude the proof of the theorem it suffices to prove that $\mathcal{T}$ is uniformly bounded on global atoms.

It is straightforward to check that for each ball B of radius 1, there exists a Lipschitz function φ_B on M such that $\varphi_B = 1$ on $2B$, $\varphi_B = 0$ on $M \setminus 3B$ and $\|\nabla \varphi_B\|_\infty \leq 1$ almost everywhere.

Suppose that b is a global atom supported in a ball B of radius 1. Observe that

$$\begin{aligned} \|\mathcal{T} b\|_1 &\leq \|\varphi_B \, \mathcal{T} b\|_1 + \|(1-\varphi_B) \, \mathcal{T} b\|_1 \\ &\leq \|\mathbf{1}_{3B} \, \mathcal{T} b\|_1 + \|(1-\varphi_B) \, \mathcal{T} b\|_1. \end{aligned} \tag{4.10}$$

We estimate the two summands above separately.

To estimate the first, we observe that, by Schwarz's inequality and the fact that μ is locally doubling,

$$\begin{aligned} \|\mathbf{1}_{3B} \, \mathcal{T} b\|_1 &\leq \mu(3B)^{1/2} \, \|\mathcal{T} b\|_2 \\ &\leq \sqrt{\frac{\mu(3B)}{\mu(B)}} \, |\!|\!| \mathcal{T} |\!|\!|_2 \\ &\leq C \, |\!|\!| \mathcal{T} |\!|\!|_2. \end{aligned} \tag{4.11}$$

To estimate the second summand we shall use the analytic Cheeger isoperimetric property [Ch, Thm 6.4], which states that there exists a constant C such that

$$\|f\|_1 \leq C \, \|\nabla f\|_1 \qquad \forall f \in L^1(\mu). \tag{4.12}$$

Observe that

$$\nabla\big[(1-\varphi_B) \, \mathcal{T} b\big] = -(\nabla \varphi_B) \, \mathcal{T} b + (1-\varphi_B) \int_B \nabla_1 k(\cdot, y) \, b(y) \, \mathrm{d}\mu(y).$$

Now we apply (4.12) to $(1-\varphi_B) \, \mathcal{T} b$, and the triangle inequality in the formula above, and obtain

$$\|(1-\varphi_B) \, \mathcal{T} b\|_1 \leq C \, \|\mathbf{1}_{3B} \, \mathcal{T} b\|_1 + C \int_{(2B)^c} \mathrm{d}\mu(x) \, \Big| \int_B \nabla_1 k(x,y) \, b(y) \, \mathrm{d}\mu(y) \Big|.$$

We estimate the first summand on the right hand side as in (4.11). To estimate the second, we use Tonelli's theorem and obtain that

$$\int_{(2B)^c} \mathrm{d}\mu(x) \Big| \int_B \nabla_1 k(x,y)\, b(y)\, \mathrm{d}\mu(y) \Big| \leq \int_B \mathrm{d}\mu(y)\; |b(y)| \int_{(2B)^c} |\nabla_1 k(x,y)|\; \mathrm{d}\mu(x).$$

Thus,

$$\|(1-\varphi_B)\mathcal{T}b\|_1 \leq C\, |\!|\!|\mathcal{T}|\!|\!|_2 + C_0. \tag{4.13}$$

By combining (4.11) and (4.13), we get that there exists a constant C such that

$$\|\mathcal{T}b\|_1 \leq C\, |\!|\!|\mathcal{T}|\!|\!|_2 + C_0$$

for all global atoms, as required to conclude the proof of the theorem. $\square$

Now suppose that M is a connected noncompact *unimodular* Lie group, endowed with a left invariant Riemannian metric and denote by μ the associated Riemannian measure (a constant multiple of the left Haar measure). We assume that M has spectral gap.

For each element X in the Lie algebra $\mathfrak{m}$ of M, denote by $\widetilde{X}_\ell$ and $\widetilde{X}_r$ the left invariant and the right invariant vector fields whose value at e is exactly X respectively. Write $\check{f}(z)$ for $f(z^{-1})$. It is straightforward to check that

$$\widetilde{X}_\ell \check{f} = -\big(\widetilde{X}_r f\big)\check{} \tag{4.14}$$

for all functions f in $C_c^\infty(M)$. Choose an orthonormal basis $X_1,\ldots,X_n$ of $\mathfrak{m}$ (with respect to the given Riemannian metric). Then

$$|\nabla f|\,(x) = \Big(\sum_{j=1}^n \big|\widetilde{(X_j)}_\ell f(x)\big|^2\Big)^{1/2}. \tag{4.15}$$

where $|\nabla f|\,(x)$ denotes the length of the Riemannian gradient of f at the point x.

Suppose that $\mathcal{T}$ is a left invariant operator, with kernel k; define the *convolution kernel* K of $\mathcal{T}$ by the rule

$$K(x) = k(x,e) \qquad \forall x \in M,$$

where e denotes the identity of the group M. Then

$$k(x,y) = K(y^{-1}x) \qquad \forall x,y \in M.$$

Note that k satisfies the local Hörmander condition (4.5) if and only if K satisfies the following

$$\sup_B \sup_{y\in B} \int_{(2B)^c} \big|K(y^{-1}x) - K(x)\big|\; \mathrm{d}\mu(x) < \infty,$$

where B runs over all balls of radius at most 1 centred at the identity. In the case where k is differentiable off the diagonal of $M\times M$, then K is differentiable off the identity, and it is often convenient to express the local Hörmander condition in the following form

$$\sup_{r\in(0,1]} r \int_{B(e,r)^c} \Big(\sum_{j=1}^n \big|\widetilde{(X_j)}_r K(x)\big|^2\Big)^{1/2} \mathrm{d}\mu(x) < \infty. \tag{4.16}$$

Corollary 4.8. *Suppose that M is a Lie group as above, that $\mathcal{T}$ is a left invariant linear operator, bounded on $L^2(\mu)$, and that its convolution kernel K satisfies the local Hörmander integral condition (4.16). Then $\mathcal{T}$ extends to a bounded operator from $\mathfrak{h}^1(\mu)$ to $L^1(\mu)$.*

Proof. Observe that if K satisfies (4.16), then the kernel k of $\mathcal{T}$ satisfies the local Hörmander integral condition (4.5). Hence, by [CMM1, Thm 8.2], $\mathcal{T}$ is bounded from $H^1(\mu)$ to $L^1(\mu)$.

We claim that the kernel k satisfies the inequality

$$\sup_{y\in M} \int_{B(y,2)^c} |k(x,y)| \,\mathrm{d}\mu(x) < \infty,$$

whence the desired conclusion follows by Proposition 4.5.

To prove the claim, we observe that by (4.14) and (4.15)

$$\begin{aligned}\int_{B(e,1)^c} \big|\nabla \check{K}(x^{-1})\big| \,\mathrm{d}\mu(x) &= \int_{B(e,1)^c} \Big(\sum_{j=1}^n \big|\widetilde{(X_j)}_\ell \check{K}(x^{-1})\big|^2\Big)^{1/2} \mathrm{d}\mu(x)\\ &= \int_{B(e,1)^c} \Big(\sum_{j=1}^n \big|\widetilde{(X_j)}_r K(x)\big|^2\Big)^{1/2} \mathrm{d}\mu(x),\end{aligned}$$

which is finite because K satisfies the Hörmander type condition (4.16). Since M is unimodular and $B(e,1)^c$ is invariant under the involution $x \mapsto x^{-1}$, we may conclude that $\big|\nabla\check{K}\big|$ is integrable on $B(e,1)^c$. Denote by φ a smooth cutoff function, which is equal to 1 in $\overline{B}(e,1/2)$, and equal to 0 in $B(e,1)^c$. Clearly $\big|\nabla[(1-\varphi)\check{K}]\big|$ is in $L^1(\mu)$ by (4.16), because K is differentiable off the origin. Then the Cheeger analytic isoperimetric inequality (4.12) implies that $(1-\varphi)\check{K}$ is in $L^1(\mu)$, so that

$$\int_{B(e,1)^c} \big|\check{K}\big| \,\mathrm{d}\mu < \infty.$$

By the unimodularity of M we may then conclude that

$$\int_{B(e,1)^c} |K| \,\mathrm{d}\mu < \infty.$$

The claim follows directly from this and the fact that $k(x,y) = K(y^{-1}x)$. This concludes the proof. □

Remark 4.9. It may be worth observing that the proof of Corollary 4.8 does not make use of Theorem 4.7, but only of the isoperimetric inequality and the fact that condition (4.16) implies the $H^1(\mu)$-$L^1(\mu)$ boundedness of $\mathcal{T}$. Note also that (4.9) would follow from a condition similar to (4.16), but with left invariant vector fields instead of right invariant vector fields. However, this condition would not imply the $H^1(\mu)$-$L^1(\mu)$ boundedness of $\mathcal{T}$.

References

[CMM1] A. Carbonaro, G. Mauceri and S. Meda, H^1 and BMO on certain measured metric spaces, `arXiv:0808.0146v1 [math.FA]`, to appear in *Ann. Scuola Norm. Sup. Pisa*.

[CMM2] A. Carbonaro, G. Mauceri and S. Meda, H^1 and BMO for certain locally doubling metric measure spaces of finite measure, `arXiv:0811.0100v1 [math.FA]`, to appear in *Colloq. Math.*

[Ch] I. Chavel, *Riemannian geometry: a modern introduction*, Cambridge University Press, 1993.

[Ch1] I. Chavel, *Isoperimetric inequalities. Differential geometric and analytic perspectives*, vol. 145 of Cambridge Tract in Mathematics, Cambridge University Press, 2001.

[CG] T. Coulhon and A. Grigoryan, Random walks on graphs wilth regular volume growth, *Geom. Funct. Anal.* **8** (1998), 656–701.

[CW] R.R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[DV] O. Dragicevic and A. Volberg, Bellman functions and dimensionless estimates of Littlewood–Paley type, *J. Oper. Theory* **56** (2006), 167–198.

[FGS] E.B. Fabes, C. Gutiérrez and R. Scotto, Weak type estimates for the Riesz transforms associated with the Gaussian measure, *Rev. Mat. Iberoamericana* **10** (1994), 229–281.

[FoS] L. Forzani and R. Scotto, The higher order Riesz transforms for Gaussian measure need not be weak type (1,1), *Studia Math.* **131** (1998), 205–214.

[FTN] A. Figà-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees.* London Math. Society Lecture Notes Series, **162**, Cambridge University Press, Cambridge, U. K., 1991.

[GMST1] J. Garcia-Cuerva, G. Mauceri, P. Sjögren and J.L. Torrea, Higher order Riesz operators for the Ornstein–Uhlenbeck semigroup, *Pot. Anal.* **10** (1999), 379–407.

[GMST2] J. Garcia-Cuerva, G. Mauceri, P. Sjögren and J.L. Torrea, Spectral multipliers for the Ornstein–Uhlenbeck semigroup, *J. D'Analyse Math.* **78** (1999), 281–305.

[GMMST1] J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, and J. L. Torrea, Functional Calculus for the Ornstein-Uhlenbeck Operator, *J. Funct. Anal.* **183** (2001), no. 2, 413–450.

[GMMST2] J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, J.L. Torrea, Maximal operators for the Ornstein–Uhlenbeck semigroup, *J. London Math. Soc.* **67** (2003), 219–234.

[G] D. Goldberg, A local version of real Hardy spaces, *Duke Math. J.* **46** (1979), 27–42.

[GST] C.E. Gutiérrez, C. Segovia and J.L. Torrea, On higher order Riesz transforms for Gaussian measures, *J. Fourier. Anal. Appl.* **2** (1996), 583–596.

[GU] C.E. Gutiérrez and W. Urbina, Estimates for the maximal operator of the Ornstein–Uhlenbeck semigroup, *Proc. Amer. Math. Soc.* **113** (1991), no. 1, 99–104.

[Gun] R.F. Gundy, Sur les transformations de Riesz pour le semigroupe d'Ornstein–Uhlenbeck , *C. R. Acad. Sci. Paris Sci. Ser. I Math.* **303** (1986), 967–970.

[Gut] C. Gutiérrez, On the Riesz transforms for Gaussian measures, *J. Funct. Anal.* **120** (1994), 107–134.

[M1] P.A. Meyer, *Transformations de Riesz pour le lois Gaussiennes*, Springer Lecture Notes in Mathematics **1059** (1984), 179–193.

[MM] G. Mauceri and S. Meda, BMO and H^1 for the Ornstein–Uhlenbeck operator, *J. Funct. Anal.* **252** (2007), 278–313.

[MMS] G. Mauceri, S. Meda and P. Sjögren, Sharp estimates for the Ornstein–Uhlenbeck operator, *Ann. Sc. Norm. Sup. Pisa, Classe di Scienze*, Serie IV, (2004), n. 3, 447–480.

[MSV] S. Meda, P. Sjögren and M. Vallarino, On the H^1–L^1 boundedness of operators, *Proc. Amer. Math. Soc.* **136** (2008), 2921–2931.

[MPS1] T. Menárguez, S. Pérez and F. Soria, The Mehler maximal function: a geometric proof of the weak type 1, *J. London Math. Soc.* (2) **61** (2000), 846–856.

[M] P.A. Meyer, *Note sur le processus d'Ornstein–Uhlenbeck*, Springer Lecture Notes in Mathematics **920** (1982), 95–132.

[Mu] B. Muckenhoupt, Hermite conjugate expansions, *Trans. Amer. Math. Soc.* **139** (1969), 243–260.

[Pe] S. Pérez, The local part and the strong type for operators related to the Gauss measure, *J. Geom. Anal.* **11**, no. 3, 491–507.

[P] G. Pisier, *Riesz transforms: a simpler analytic proof of P.A. Meyer's inequality*, Springer Lecture Notes in Mathematics **1321** (1988), 485–501.

[PS] S. Pérez and F. Soria, Operators associated with the Ornstein-Uhlenbeck semigroup, *J. London Math. Soc.* **61** (2000), 857–871.

[S] P. Sjögren, *On the maximal function for the Mehler kernel*, in *Harmonic Analysis, Cortona*, 1982, Springer Lecture Notes in Mathematics **992** (1983), 73–82.

[St1] E.M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Annals of Math. Studies, No. **63**, Princeton N. J., 1970.

[St2] E.M. Stein, *Harmonic Analysis. Real variable methods, orthogonality and oscillatory integrals*, Princeton Math. Series No. **43**, Princeton N. J., 1993.

[T] M.E. Taylor, Hardy spaces and bmo on manifolds with bounded geometry, *J. Geom. Anal.* 19 (2009), no. 1, 137–190.

[U] W. Urbina, On singular integrals with respect to the Gaussian measure, *Ann. Sc. Norm. Sup. Pisa, Classe di Scienze, Serie IV*, **XVIII** (1990), no. 4, 531–567.

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