Simplex Sliding Mode Control of Multi-Input Systems with Chattering Reduction and Mono-Directional Actuators ⋆

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Abstract

This paper analyzes features and problems related to the application of the simplex sliding mode control to systems with mono-directional actuators and integrators in the input channel. The plain use of the method formulated in previous contributions results in unacceptable behaviors, such as control laws (the output of the mono-directional actuators), which increase without bounds. It is proposed a non trivial modification of the original algorithm; the new simplex strategy allows the perfect fulfillment of the control objectives by means of bounded inputs from the actuators.

Key words: Simplex sliding mode control; Multi-input systems; Chattering reduction; Uncertain dynamic systems.

1 Introduction

The simplex sliding mode control methodology dates back to the pioneering work of Bajda and Izosimov, [1], and it was extensively analyzed in previous papers, [6], [7].

In the recent work [7] a class of uncertain nonlinear non affine control systems has been dealt with by this control strategy applied to the first time derivative of the actual control vector. In principle this "trick" (negligible dynamics of actuators and sensors) counteracts the chattering phenomenon, which is often considered the main drawback of the sliding mode control methodology, in practice it allows to reduce the phenomenon to an acceptable, high frequency, small amplitude perturbation of the ideal control law.

In many real situations, either by virtue of the physical principle underlying the control action (e.g. jets, tendons, etc), or due to the presence of unilateral constraints (e.g. contact forces for manipulation, locomotion, etc), it is necessary to consider, as a further constraint on the design of the control law, the condition that any actuator can exert its action in only one direction. This paper is focused on the problem of implementing simplex sliding mode control laws in the first time derivative of the control vector by a mono-directional actuation system. In general multi-input situations any actuator generates a control vector characterized by a direction (specified by a unit vector) and an intensity (the norm of the control vector). A system with mono-directional control devices is fully actuated if any generic control law can be generated as a nonnegative linear combination of the mono-directional vectors (e.g. the force closure conditions in robotic manipulation). Full actuation with minimum number of devices implies that

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the associated the mono-directional vectors form a simplex in the relevant input space. Moreover, if the simplex of the actuation coincides with the one chosen for the control algorithm [7], just one device at a time has a first time derivative different from zero.

In this paper we show that the plain application of the simplex logic presented in [7] with mono-directional actuators would lead to a pathological behavior, that is while the designed cumulative control action is bounded, the effort required to any single device increases without bound. This phenomenon is prevented by introducing a modified simplex switching logic, which is proven to simultaneously guarantee the achievement of the desired sliding motion and the boundedness of the action exerted by any control device in a bounded domain.

Throughout the paper a prime denotes transpose and $|\cdot|$ is the Euclidean norm or the induced matrix norm.

2 Problem Statement and Previous Results

Consider the control system

$$\dot{x} = f(t, x, u), \qquad t \geq 0, \tag{1}$$

with the control vector $u \in R^K$, the state variable $x \in R^N$ and the dynamics $f : [0, +\infty) \times R^N \times R^K \to R^N$.

The control objective is to steer the trajectories of system (1) onto the sliding manifold

$$s(t, x) = 0, \tag{2}$$

where the sliding output $s(t, x)$, $s : [0, +\infty) \times R^N \to R^M$, $M \leq N$, is suitably chosen to guarantee desired behaviors of the system.

The first control constraint, which must be satisfied, is to design and apply to system (1) continuous vectors u.

To this end, according to a standard chattering reduction strategy, we consider the control u generated by simplified actuation dynamics and define the augmented control system

$$\dot{x} = f(t, x, u), \qquad \dot{u} = v, \qquad t \geq 0, \tag{3}$$

with augmented state variable $y = (x', u')' \in R^{N+K}$, control variable $v \in R^K$, and dynamics $g(t, y, v) = (f(t, y), v)'$. It is assumed that $K = M$. If the control v is discontinuous, then the vector u turns out to be continuous.

We define a new sliding output

$$\sigma(t, x, u) = \dot{s} + \Lambda s, \tag{4}$$

where $\Lambda = \mathrm{diag}(\lambda_i)$, $\lambda_i > 0$, $i = 1, \ldots, M$, $\dot{s} = D(t, x, u)$, $\ddot{s} = C(t, x, u) + B(t, x, u)\,\dot{u}$; it is assumed that f, s are both of class C^2 everywhere.

Then for almost every t

$$\dot{\sigma}(t, x, u, \dot{u}) = A(t, x, u) + B(t, x, u)\,\dot{u}, \tag{5}$$

where $A = \sigma_t + \sigma_x f = C + \Lambda D$, $B = \sigma_u = s_x f_u$ and A, B are continuous.

The relative degree between the sliding output σ and the control vector v is uniformly one.

The aim is to control the state variables $y(t) = (x'(t), u'(t))'$, $t \geq 0$, of the augmented system in order to guarantee the sliding property

$$\sigma[t, x(t), u(t)] = 0 \tag{6}$$

for every t sufficiently large. If $\sigma = 0$, the original sliding output $s\left[t, x\left(t\right)\right] \to 0$ as $t \to +\infty$ exponentially fast. This means that $s\left[t, x\left(t\right)\right]$ is arbitrarily close to 0 for t sufficiently large.

The problem is solved by the following simplex control strategy, proposed in [7].

System (3) and sliding outputs s and σ must satisfy the following assumptions.

For every (t, x, u) the uncertain control matrix B can be expressed as

$$B\left(t, x, u\right) = \overline{B}\left(t, x, u\right) + \Delta B\left(t, x, u\right), \tag{7}$$

both the uncertain control matrix B and the known nominal matrix $\overline{B}$ are everywhere non-singular.

A known constant γ_0 is available to the controller such that

$$\left|\Delta B\, \overline{B}^{-1}\right| \leq \gamma_0. \tag{8}$$

A known function γ is available to the controller such that

$$\left|C\right| \leq \gamma\left(t, x, u\right). \tag{9}$$

We consider uncertain systems, then $\dot{s}$, and therefore σ, are not available to the controller. We make up for this lack of information with a second order sliding mode observer, [11], [2], based on the so called second order sliding mode suboptimal algorithm [5],

$$\ddot{z} = w + \overline{B}\left(t, x, u\right)\dot{u}, \qquad z, w \in R^M. \tag{10}$$

The observer control vector w, designed by a decoupled second order sliding mode control law, [7], is able to globally steer to zero in finite time the observation error vectors $e = z - s$ and $\dot{e} = \dot{z} - \dot{s}$. The lack of a priori known constant bounds of systems uncertainties prevent the use of different second order sliding mode observers [11], [8], which exploit some homogeneity property [14], [12], [13] of the differential inclusions representing the uncertain discontinuous error equation.

In order to design a simplex control strategy based on the estimate of $\dot{s}$, provided in finite time by (10), fix $M+1$ constant vectors $p_1, \ldots, p_{M+1} \in R^M$ such that $\left|p_i\right| = 1$, $i = 1, \ldots, M+1$, and there exists a constant $c \neq 0$ such that they satisfy the following obtuse angle condition, [4], $p_i' p_h \leq -c^2 \left|p_i\right|\left|p_h\right| \leq -c^2$, $i \neq h$. They form a simplex of vectors in R^M.

Let the $M+1$ vectors $v_1, \ldots, v_{M+1} \in R^M$ be defined as $v_i = a\left(t\right) p_i$, $i = 1, \ldots, M+1$, where the function $a\left(t\right)$ is integrable on every bounded interval of $[0, +\infty)$ and $a\left(t\right) > 0$ for any $t \geq 0$. The vectors v_i, $i = 1, \ldots, M+1$, form a simplex and satisfy for every $t \geq 0$

$$\begin{aligned} &v_i' v_h \leq -c^2 \left|v_i\right|\left|v_h\right| \quad \text{if} \quad i \neq h, \quad \text{and} \\ &\quad 0 < a\left(t\right) = \left|v_i\right|, \quad i = 1, \ldots, M+1. \end{aligned} \tag{11}$$

The space R^M is partitioned in $M+1$ cones

$$\begin{aligned} Q_h = Q_h\left(t\right) = \operatorname{cone}\left(v_i : \ i = 1, \ldots, M+1, \ i \neq h\right), \\ h = 1, \ldots, M+1, \end{aligned} \tag{12}$$

with pairwise disjoint interiors.

Define the new vector

$$\hat{\sigma} = \dot{z} + \Lambda z \tag{13}$$

where $\Lambda = \text{diag}(\lambda_i)$, $i = 1, \ldots, M$, is the same as in (4) and $\max\limits_{i=1,\ldots,M} \lambda_i = \lambda_{max}$.

Given $(t, x, u, z, \dot{z}) \in [0, +\infty) \times R^N \times R^M \times R^M \times R^M$, let h be the least index such that, by (12),

$$\hat{\sigma}(t, x, u, z, \dot{z}) \in Q_h.$$

The simplex control algorithm based on estimates is defined by the discontinuous switching logic:

$$\begin{array}{l} \text{if} \quad \hat{\sigma}(t, x, u, z, \dot{z}) \in Q_h \quad \text{then} \\ \quad v^*(t, x, u, z, \dot{z}) = \overline{B}^{-1}(t, x, u)\, v_h \end{array} \tag{14}$$

and v^* is the simplex control law based on estimates.

The control w of the estimation process guarantees that, in finite time, the estimation errors e and $\dot{e}$ are zero. The switching logic (14) is applied with

$$a(t) \geq \frac{|w| + \lambda_{max}|\dot{z}| + k^2}{c^2}, \quad k \neq 0, \quad t \geq 0. \tag{15}$$

In this case, Theorem 2 in [7] assures that the actual sliding output σ converges to the sliding manifold $\sigma = 0$ in finite time.

3 Nonlinear Systems with Mono-Directional Actuators

In this section we consider nonlinear systems actuated by mono-directional devices. Many physical systems are controlled by this kind of actuators, each one of which generates an action which can be continuously modulated only in one direction (contact forces, tendons, solenoids, jets, etc.).

Since mono-directional actuation is taken into account, it is necessary to consider a further constraint on the design of the control vectors according to the simplex sliding mode strategy.

Assume that the nonlinear uncertain control system (3) is actuated by P mono-directional devices. The control vector u is generated as the non-negative linear combination of some vectors $h_i \in R^M$, $i = 1, \ldots, P$, directly related to the P actuators and their disposition

$$u = \sum_{i=1}^{P} \overline{u}_i = \sum_{i=1}^{P} h_i F_i(t) = HF(t), \quad F_i \geq 0, \tag{16}$$

where $\overline{u}_i = h_i F_i$, $\forall i \in \{1, \ldots, P\}$; the vectors h_i, $i = 1, \ldots, P$, express the directions along which the actuators exert their positive manipulable actions, the intensities of which are measured by the quantities $F_i(t)$, $i = 1, \ldots, P$; the matrix $H \in R^{M \times P}$ is defined as $H = \text{col}(h_i)$ and the vector $F \in R^P$ is given by $F = [F_1, \ldots, F_P]'$.

The fact that the $F_i(t)$, $i = 1, \ldots, P$, are non-negative, derives from the mono-directionality of the actuators. The matrix H could depend on time and state variables, but in this paper it is assumed to be constant.

In order to have a fully actuated system, the choice, if allowed, which minimizes the required number of mono-directional devices, is to consider $P = M+1$ actuators disposed in a way that the columns h_i, $i \in \mathcal{A} = \{1, \ldots, M+1\}$ of H form a simplex such that for some constant $c \neq 0$

$$|h_i| = 1, \quad h_i' h_j \leq -c^2, \quad \forall i, j \in \mathcal{A},\ i \neq j. \tag{17}$$

It follows that the $M + 1$ vectors

$$\overline{u}_i = h_i F_i, \quad \forall i \in \mathcal{A}, \tag{18}$$

form a simplex of vectors, which satisfy

$$\begin{aligned}
&|\overline{u}_i| = F_i, \quad \overline{u}_i' \overline{u}_j \leq -c^2 F_i F_j, \\
&F_i,\, F_j \geq 0, \quad \forall i, j \in \mathcal{A},\, i \neq j.
\end{aligned}$$

This fact guarantees that any control vector $u \in R^M$ can be generated.

Let us differentiate (16) and remember $\dot{u} = v$, we obtain

$$v = \sum_{i=1}^{M+1} \overline{v}_i = \sum_{i=1}^{M+1} h_i \dot{F}_i(t) = H\dot{F}(t), \tag{19}$$

where $\dot{F} = \left[\dot{F}_1, \ldots, \dot{F}_{M+1}\right]'$.

Consider system (3), the control vector v, the sliding output $s(t,x)$, the sliding vector $\sigma(t,x,u)$ defined by (4) and (5). Suppose that assumptions (7)–(9) hold.

From (17), if in (19) we set $\dot{F}_i(t) = a(t)$, $i \in \mathcal{A}$, with $a(t) \geq 0$, we have that the $M+1$ vectors $\overline{v}_i$, $i \in \mathcal{A}$, form a simplex of vectors in R^M corresponding to (11).

Suppose that the sliding output $\hat{\sigma} = \sigma + \eta$ is available. We apply the control $\tilde{v}(t)$ such that

$$\text{if} \qquad \hat{\sigma} \in Q_h \qquad \text{then} \qquad \tilde{v}(t) = \overline{v}_h. \tag{20}$$

If the sliding observer (10) is used, $\eta = 0$ in finite time. It follows by Theorem 2 in [7] that the feedback control $\tilde{v}$ (20) guarantees that, provided the positive control gain $a(t)$ is chosen according to the condition (15), for every t sufficiently large, the state of the control system (3) satisfies the sliding property (6).

We need conditions giving boundedness of the control vector guaranteeing $\sigma = 0$.

Proposition 1 *Let $\sigma[t, x(t), u(t)] = 0$ and $|x(t)| \leq L$ for all $t \geq \overline{t}$. Write*

$$g(t,y,w) = \frac{\partial s(t,y)}{\partial t} + \frac{\partial s(t,y)}{\partial x} f(t,y,w).$$

Then u is bounded provided

$$\liminf_{w\to\infty} \ \inf\left\{|g(t,y,w)| \ : \ t \geq \overline{t}, |y| \leq L\right\} \neq 0. \tag{21}$$

Proof of Proposition 1.

Arguing by contradiction, let u be unbounded. Since u is continuous, there exists a sequence $t_n \to +\infty$ such that $u(t_n) \to \infty$. Then by (21)

$$\begin{aligned}
0 &\neq \liminf_{w\to\infty} \ \inf\left\{|g(t,y,w)| : \ t \geq \overline{t}, |y| \leq L\right\} \\
&\leq \liminf_{n\to+\infty} \ \inf\left\{|g[t,y,u(t_n)]| : \ t \geq \overline{t}, |y| \leq L\right\} \\
&\leq \liminf_{n\to+\infty} |g[t_n, x(t_n), u(t_n)]|.
\end{aligned} \tag{22}$$

By (6) we have

$$|\dot{s}(t_n)| = |g[t_n, x(t_n), u(t_n)]| \leq (\text{const.})\, \mathrm{e}^{\alpha t_n}$$

with $\alpha < 0$. Then by (22) we get a contradiction. $\square$

We remark that (21) is true in the special case of

$$f(t,x,u) = Ax + Bu, \quad s(t,x) = Cx$$

with A, B, C constant matrices, provided the rank of CB is K.

Because of its discontinuous nature, the vector $\dot{u} = v$ commutes at infinite frequency among a finite number of vectors $\overline{v}_i = h_i \dot{F}_i(t) = h_i a(t)$. There exists at least one $\dot{F}_i$, which oscillates at infinite frequency between zero and positive values $a(t) \geq \frac{k^2}{c^2}$ from (15); it follows that at least one $F_i(t)$ increases without bounds, as $t \to +\infty$.

Considering this fact, the following proposition proves that the application of the simplex method as previously formulated, leads to an unacceptable behaviour, since any actuator intensity $F_i(t)$ tends to increase without bounds.

Proposition 2 *Suppose that (21) holds. On $\sigma = 0$ if there exists an index J, $J \in \mathcal{A}$, such that F_J tends to infinity, then all the F_i, $i \in \mathcal{A}$, tend to infinity.*

If, on the contrary, there exists an index I, $I \in \mathcal{A}$, such that F_I is bounded, then all the F_i, $i \in \mathcal{A}$, are bounded.

Proof of Proposition 2.

The sliding condition $\sigma = 0$ is fulfilled after a finite time t_2. Therefore, for any $t \geq t_2$, it exists a control vector

$$u(t) = u_\sigma(t) \quad \text{such that} \quad \sigma(t, x, u_\sigma) = 0,$$

u_σ bounded according to Proposition 1.

As any other vector in R^M, $u_\sigma(t)$ can be expressed as the linear combination with non-negative coefficients, of M vectors h_i of the simplex

$$u_\sigma(t) = \sum_{i=1, i \neq h}^{M+1} h_i \lambda_i(t), \qquad \lambda_i(t) \geq 0. \tag{23}$$

where $\lambda_i(t)$ are bounded functions.

Recalling (18), we can write $u_\sigma(t)$ as follows

$$u_\sigma(t) = \sum_{i \in \mathcal{A}} h_i F_i, \tag{24}$$

where the set $\mathcal{A} = \{1, \ldots, M+1\}$ contains all the $M+1$ indexes of the vectors of the simplex.

Comparing (23) and (24) we obtain

$$HF = H_{\mathcal{D}} \lambda \tag{25}$$

where the matrix $H \in R^{M \times (M+1)}$ is full rank and $H_{\mathcal{D}}$ is the matrix, the columns of which are the vectors of the simplex h_i, $i \in \mathcal{D}$, where the set $\mathcal{D}$ is such that $\mathcal{A} \setminus \mathcal{D}$ is a singleton.

Expression (25) represents M linear equations in $M+1$ variables, where H is the matrix of coefficients, F is the column vector of variables, and $H_{\mathcal{D}} \lambda$ is the column vector of solutions.

System (25) is underdetermined since H is full rank; then any solution of (25) can be expressed as the sum of a particular solution $F^* \in R^{M+1}$ and of a vector belonging to $\mathrm{null}\,(H)$.

A particular solution of (25) is given by

$$F_i^* = \begin{cases} \lambda_i, & i \in \mathcal{D}, \\ 0, & i \in \mathcal{A} \setminus \mathcal{D}. \end{cases}$$

The columns of H form a simplex then there exist constants μ_i, $i \in \mathcal{A}$, such that

$$\sum_{i\in\mathcal{A}} \mu_i h_i = 0, \qquad \mu_i > 0, \quad \forall i, \qquad \text{and} \qquad \sum_{i\in\mathcal{A}} \mu_i = 1.$$

Let $\mu_H \in R^{M+1}$ be the constant vector such that $\mu_{H\,i} = \mu_i$, $i \in \mathcal{A}$, therefore $F = F^* + k\mu_H$, $k \in R$, that is

$$F_i = F_i^* + k\mu_i = \begin{cases} \lambda_i + k\mu_i, & i \in \mathcal{D}, \\ k\mu_i, & i \in \mathcal{A} \setminus \mathcal{D}. \end{cases} \tag{26}$$

From (26), if there exists an index J such that F_J tends to infinity, then k tends to infinity and all the F_i, $i \in \mathcal{A}$, tend to infinity.

From (26) it is also apparent that if for some reason it is possible to ensure that there exists an index I such that F_I is bounded, then k is bounded and all the F_i, $i \in \mathcal{A}$, are bounded. $\square$

We introduce a modified switching logic, which guarantees that σ is steered to zero in finite time while all the F_i, $i \in \mathcal{A}$, remain bounded. That is, according to Proposition 2, $\min\limits_{i\in\mathcal{A}} F_i \leq F_0$, where F_0 is an arbitrarily fixed finite constant value.

The simplex switching logic is modified taking into account the fact that while a mono-directional action F_i, $i \in \mathcal{A}$, cannot assume negative values, its derivative $\dot{F}_i$, $i \in \mathcal{A}$, can be made negative, which corresponds to a decreasing F_i. Within a fixed range $0 < F_i \leq F_{Max}$, $i \in \mathcal{A}$, any mono-directional actuator can generate a bidirectional action in terms of $\dot{F}_i$.

In (20) any control vector $\overline{v}_i$, $i \in \mathcal{A}$, is given by

$$\overline{v}_i = h_i \dot{F}_i\,(t) = He_i a\,(t)\,, \tag{27}$$

where e_i is the i-th vector of the standard orthonormal basis of R^{M+1}. The feedback control $v_*\,(t)$, either $\tilde{v}\,(t)$, is generated by the mono-directional actuators, such that just one device at a time has a first time derivative different from zero

The columns of H form a simplex then $\mathrm{null}\,(H) = \mathrm{span}\,(\mu_H)$, where $\mu_H \in R^{M+1}$ is the constant vector such that $\mu_{H\,i} = \mu_i$, $i \in \mathcal{A}$; the constant elements μ_i, $i \in \mathcal{A}$, are such that

$$\sum_{i\in\mathcal{A}} \mu_i h_i = 0, \qquad \mu_i > 0, \quad \forall i, \qquad \text{and} \qquad \sum_{i\in\mathcal{A}} \mu_i = 1.$$

The vectors $\overline{v}_i\,(t)$, $i \in \mathcal{A}$, in (27), can be rewritten as

$$\begin{aligned} \overline{v}_i\,(t) &= h_i \dot{F}_i\,(t) - b\,(t) \textstyle\sum_{i\in\mathcal{A}} \mu_i h_i = \\ &= He_i a\,(t) - H\mu_H b\,(t) = \\ &= H\beta_i\,(t)\,, \end{aligned} \tag{28}$$

where $b(t) \in R$ is a positive scalar and e_i is the i-th unit vector of the standard orthonormal basis of R^{M+1}.

The vectors $\beta_i(t) \in R^{M+1}$, $i \in \mathcal{A}$, are such that

$$\begin{aligned} &\beta_{ij}(t) = -b(t)\,\mu_j \quad i \neq j \quad \text{and} \\ &\beta_{ii}(t) = -b(t)\,\mu_i + a(t) \quad i, j \in \mathcal{A}. \end{aligned}$$

If $b(t)\min_{i\in\mathcal{A}} \mu_i > a(t) > 0$, then each component of $\beta_i(t)$, $i \in \mathcal{A}$, turns out to be negative.

Repeating the reasoning, vectors $\overline{v}_i(t)$, $i \in \mathcal{A}$, in (27), can be rewritten as

$$\begin{aligned} \overline{v}_i(t) &= h_i \dot{F}_i(t) + d(t) \textstyle\sum_{i\in\mathcal{A}} \mu_i(t)\, h_i = \\ &= He_i a(t) + H\mu_H d = \\ &= H\delta_i(t), \end{aligned} \tag{29}$$

where $d(t) \in R$ is a positive scalar.

The vectors $\delta_i(t) \in R^{M+1}$, $i \in \mathcal{A}$, are such that

$$\begin{aligned} &\delta_{ij}(t) = d(t)\,\mu_j \quad i \neq j \quad \text{and} \\ &\delta_{ii}(t) = d(t)\,\mu_i + a(t) \quad i, j \in \mathcal{A}. \end{aligned}$$

Since $d(t) > 0$, each component of $\delta_i(t)$, $i \in \mathcal{A}$, turns out to be positive.

The three equivalent forms (27), (28) and (29), in which the control vectors $\overline{v}_i(t)$, $i \in \mathcal{A}$, can be expressed, correspond to three possible status of activation of the actuators, which produce the same effect.

The Simplex Algorithm with Switching Logic for Bounded Mono-Directional Control Actions

R^M is partitioned in $M+1$ cones

$$Q_h = Q_h(t) = \operatorname{cone}(\overline{v}_i(t) : \ i = 1, \ldots, M+1, \ i \neq h), \qquad h = 1, \ldots, M+1,$$

with pairwise disjoint interiors.

Given $t \geq 0$, the control algorithm is defined by the discontinuous switching logic

$$\text{if} \quad \hat{\sigma} \in Q_h \quad \text{then}$$

$$\underline{v}(t) = \overline{v}_h(t) = \begin{cases} H\delta_h(t), \text{ if } \left(F_0 - \min_{i\in\mathcal{A}} F_i(t)\right) > 0, \\[2ex] H\beta_h(t), \text{ if } \left(F_0 - \min_{i\in\mathcal{A}} F_i(t)\right) \leq 0. \end{cases} \tag{30}$$

where h is the least index such that $\hat{\sigma}(t) \in Q_h$, and F_0 is a prescribed constant.

The following theorem is proven.

Theorem 1 *Consider system (3), the control vector v, the sliding output $s(t, x)$, the sliding vector $\sigma(t, x, u)$ defined by (4), (5) and the vector $\hat{\sigma}$ defined by (13). Suppose that assumptions (7)–(9) hold.*

Then there exists a time instant t_3 such that, for any $t \geq t_3$, every uncertain state corresponding to $\underline{v}(t)$ verifies the sliding condition $\sigma = 0$ and $\min\limits_{i\in\mathcal{A}} F_i = F_0$.

The proof of Theorem 1 requires the following lemma [15].

Lemma 1 *Let*

$$v(y) = \max\{f_1(y), \ldots, f_p(y)\}$$

where $f_1, \ldots, f_p$ are of class C^1. Let $I(y)$ the set of the active indices, i.e.

$$i \in I(y) \text{ iff } f_i(y) = v(y).$$

Then, if the function x is differentiable at t, we have

$$\frac{d}{dt} v[x(t)] = z' \dot{x}(t), \quad \forall z \in \partial v[x(t)],$$

and moreover [10]

$$\partial v(y) = \text{co}\{\nabla f_i(y) : \ i \in I(y)\}.$$

Proof of Theorem 1.

From Theorem 2 in [7], we have that $\sigma = 0$ in finite time. In order to show that $\min\limits_{i\in\mathcal{A}} F_i = F_0$, let us consider the following quantity

$$\begin{aligned} \psi(F) &= F_0 - \min_{i\in\mathcal{A}} F_i = \\ &= \max\nolimits_{i\in\mathcal{A}} [(F_0 - F_i)] = \\ &= \max[(F_0 - F_1), \ldots, (F_0 - F_{M+1})]. \end{aligned}$$

According to Lemma 1, we have

$$\partial\psi(F) = \text{co}[(-e_i), \ i \in I(F)] = -\sum_{i\in I(F)} \rho_i e_i,$$

where the vectors $e_i \in R^{M+1}$, $i \in \mathcal{A}$, form the standard orthonormal basis of R^{M+1}, $\rho_i \geq 0$, $i \in I(F)$, $\sum\limits_{i\in I(F)} \rho_i = 1$, and

$$\dot{\psi}(F) = \partial\psi(F)\dot{F} = -\sum_{i\in I(F)} \rho_i e_i' \dot{F}. \tag{31}$$

Let us apply the switching logic (30)

$$\text{if} \qquad \hat{\sigma} \in Q_h \qquad \text{then}$$

$$\underline{v} = \overline{v}_h = \begin{cases} H\delta_h, \text{ if } \left(F_0 - \min\limits_{i\in\mathcal{A}} F_i\right) > 0, \\ \\ H\beta_h, \text{ if } \left(F_0 - \min\limits_{i\in\mathcal{A}} F_i\right) \leq 0. \end{cases}$$

Considering (31) and (30), two cases can occur.

Case 1: $\psi > 0$, then $\dot{\psi} = -\sum_{i \in I(F)} \rho_i \delta_{hi} < 0$, since $\rho_i \geq 0$, $i \in I(F)$, $\sum_{i \in I(F)} \rho_i = 1$ and $\delta_{hi} > 0$, $h \in \mathcal{A}$, $i \in I(F)$.

Case 2: $\psi \leq 0$, then $\dot{\psi} = -\sum_{i \in I(F)} \rho_i \beta_{hi} > 0$, since $\rho_i \geq 0$, $i \in I(F)$, $\sum_{i \in I(F)} \rho_i = 1$ and $\beta_{hi} < 0$, $h \in \mathcal{A}$, $i \in I(F)$.

In both cases, we see that

$$\psi \dot{\psi} \leq -\delta |\psi|, \quad \delta > 0,$$

so that $\psi = 0$ in finite time. $\square$

Remark 1 *The proposed procedure guarantees almost the same robustness and tracking precision of the standard simplex sliding mode control, together with the chattering elimination. Moreover this methodology introduces the possibility to control the power consumption. Indeed the on-off logic of the standard simplex method dissipates a power which is equivalent to that of a single actuator at its maximum power even if the equivalent control is zero. With the proposed control we can choose F_0 as close to zero as we want, attaining a control signal which is arbitrarily close to the equivalent control [16]. The equivalent control can be characterized by some optimality property if the sliding manifold is suitably chosen.*

4 Example: Simplex Based Maneuvering of a Surface Vessel

We consider, as an example, the problem of controlling the planar position and orientation of an autonomous surface vessel using four mono-directional propellers, Figure 1.

The surface vessel has three degrees of freedom (DOF), which are the position of the center of mass (x_I, y_I) and the heading ψ of the vehicle in the earth-fixed inertial frame (I-frame).

The dynamic model of the vessel can be expressed, [9], as

$$\begin{aligned} \dot{\eta} &= R(\psi)\,\nu, \\ M\dot{\nu} &= -C(\nu)\,\nu - D(\nu)\,\nu + \tau, \end{aligned} \tag{32}$$

where $\eta = (x_I, y_I, \psi)'$ is the position vector, $\nu = (v_x, v_y, \omega_z)'$ is the velocity vector and $\tau \in R^3$ denotes the vector of external forces and torque generated by the four mono-directional propellers. The surge and sway (v_x, v_y) are the linear velocities and the yaw ω_z is the angular rate of the vessel in the the vehicle-fixed (B-frame). For simplicity we assume that the origin of the B-frame is located at the center of mass of the system. Also we suppose that the vehicle is neutrally buoyant. $M \in R^{3\times3}$ is the positive definite inertia matrix, including added mass; $C(\nu) \in R^{3\times3}$ and $D(\nu) \in R^{3\times3}$ denote the Coriolis/centrifugal and the damping matrices, respectively. We assume that M is constant and diagonal and the hydrodynamic damping terms of order higher than one are neglected. The three DOF rotation matrix $R(\psi)$ is such that $R(\psi) R'(\psi) = I$, $|R(\psi)| = 1$ for all ψ, and $\frac{d}{dt} R(\psi) = \dot{\psi} R(\psi) S$, where

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

Let $x = (\eta', \nu')'$, $u = \tau$ and $\dot{u} = v$; the dynamics of the surface vessel (32) can be rewritten in the form (3)

$$\dot{x} = f(t, x, u), \quad \dot{u} = v, \quad t \geq 0,$$

where $f_1 = x_4 \cos x_3 - x_5 \sin x_3$, $f_2 = x_4 \sin x_3 + x_5 \cos x_3$, $f_3 = x_6$, $f_4 = \frac{m_{22}x_5x_6 - d_{11}x_4 + u_1}{m_{11}}$, $f_5 = \frac{-m_{11}x_4x_6 - d_{22}x_5 + u_2}{m_{22}}$, $f_6 = \frac{-(m_{22}-m_{11})x_4x_5 - d_{33}x_6 + u_3}{m_{33}}$, with $m_{11} = 200\,kg$, $m_{22} = 250\,kg$, $m_{33} = 80\,kg\,m^2$, $d_{11} = 70\,kg\,s^{-1}$, $d_{22} = 100\,kg\,s^{-1}$, $d_{33} = 50\,kg\,m^2\,s^{-1}$.

The system is designed, [3], such that the input vector u is generated as the non-negative linear combination of the vectors $h_i \in R^3$, $i = 1, \ldots, 4$, directly related to the four actuators and their disposition

$$u = \sum_{i=1}^{4} \overline{u}_i = \sum_{i=1}^{4} h_i F_i\left(t\right) = HF\left(t\right), \quad F_i \geq 0, \tag{34}$$

where $\overline{u}_i = h_i F_i$, $\forall i \in \{1, \ldots, 4\}$; the vector $F \in R^4$ is given by $F = [F_1, \ldots, F_4]'$ and the quantities F_i, $i = 1, \ldots, 4$, are non-negative, due to the mono-directionality of the propellers. The matrix $H \in R^{3\times 4}$, $H = \mathrm{col}\left(h_i\right)$, $i = 1, \ldots, 4$, is designed as

$$H = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 \end{bmatrix}. \tag{35}$$

The configuration matrix H relates the input vector u to the positive thrusts $F_1, \ldots, F_4$ exerted by the four propellers, Figure 1.

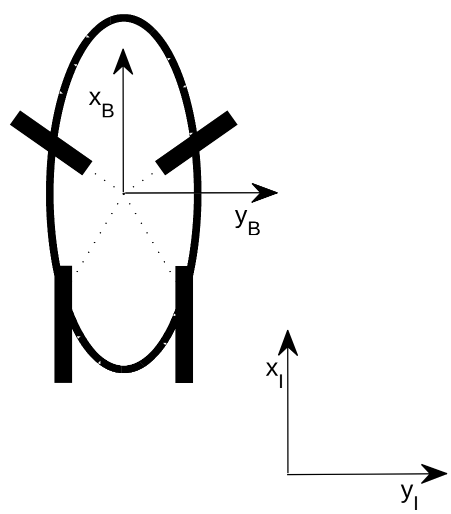

Fig. 1. The system is actuated by four mono-directional devices.

According to the proposed procedure we define the sliding manifold $s\left(t, x\right) = 0$ with $s = \nu - \nu_d$, being $\nu_d\left(t\right)$ the desired reference trajectories to be tracked, and the sliding output $\sigma\left(t, x, u\right) = \dot{s} + \Lambda s$, $\Lambda = \mathrm{diag}\left(\lambda_i\right)$, $i = 1, \ldots, 4$.

The vector $\dot{s}$ is not available. It is designed the second order sliding mode observer (10) for system (32), from which we obtain $\dot{z}$ and therefore the sliding vector $\hat{\sigma}$ defined by (13).

We apply the simplex algorithm (30) with switching logic for bounded mono-directional control actions in order to steer to zero the sliding output $\hat{\sigma}$, Figure 2. According to Theorem 1 in finite time σ is zero and s tends to zero exponentially, Figure 3. The designed F_i, $i = 1, \ldots, 4$, are continuous, bounded and controlled, since $\min\limits_{i=1,\ldots,4} F_i$ is maintained to a fixed value F_0, Figure 4.

The discontinuous $\dot{F}_i$, $i = 1, \ldots, 4$, are designed by (30) and oscillate between positive and negative values, Figure 5.

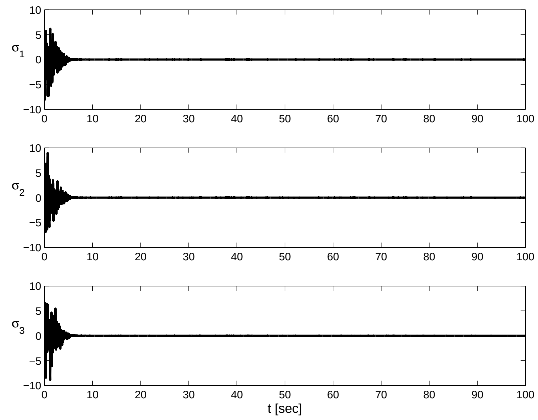

Fig. 2. The finite time convergence of $\hat{\sigma}_i$, $i = 1,\, 2,\, 3$.

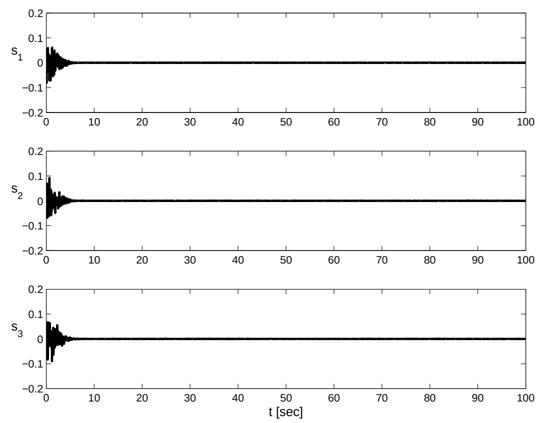

Fig. 3. The exponential convergence of s_i, $i = 1,\, 2,\, 3$.

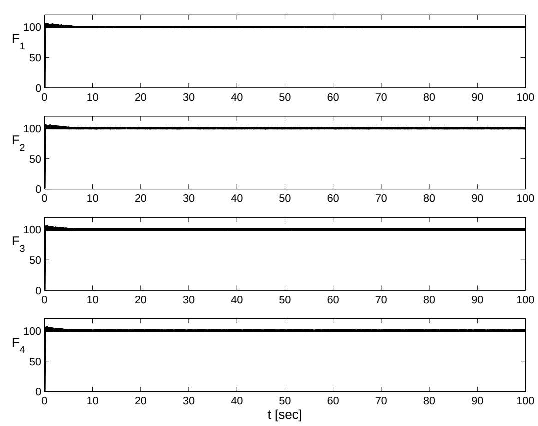

Fig. 4. The continuous, bounded and controlled F_i, $i = 1, \ldots, 4$.

The continuous input vector $u = \tau$ is the non-negative linear combination of the columns h_i, $i = 1, \ldots, 4$, of the matrix H defined by (35), that is $u = \sum_{i=1}^{4} h_i F_i\,(t)$, Figure 6.

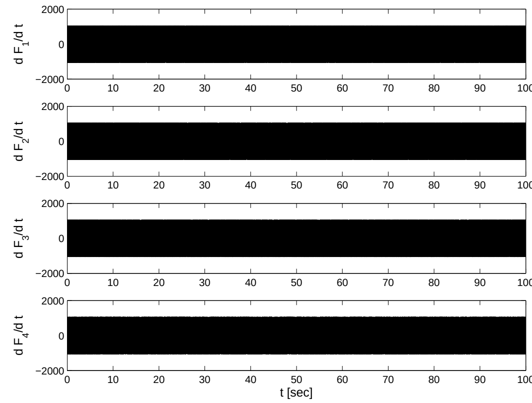

Fig. 5. The discontinuous $\dot{F}_i$, $i = 1, \ldots, 4$.

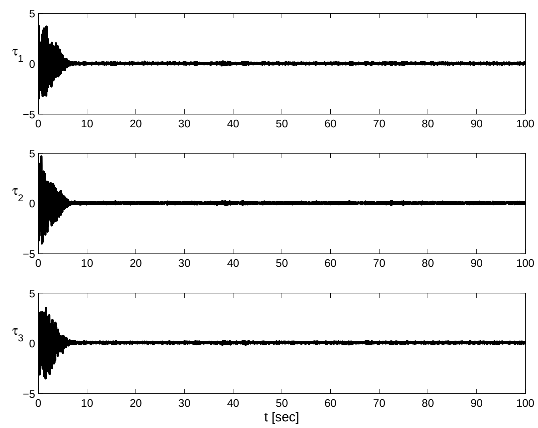

Fig. 6. The continuous $u_i = \tau_i$, $i = 1, 2, 3$.

5 Conclusions

The application of the simplex sliding mode control method to the specific case of systems with mono-directional actuators has been considered. The number of control devices turns out to be minimized if their collocation is arranged such that the corresponding control vectors form a simplex in the relevant input/output space. The introduction of integrators in the input channel, for chattering reduction purposes, has been proved to require a modification of the original methodology to avoid unbounded increment of the actuators intensity. The new control algorithm exploits the geometrical property of the simplices of vectors to attain the same tracking efficiency with different switching logics. This fact guarantees an extra degree of freedom, the exploitation of which decouples the tracking control problem of the system with that of the regulation of each actuator's intensity.

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