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DECOHERENCE FOR POSITIVE SEMIGROUPS ON $M_2(\mathbb{C})$

R. Carbone[1], **E. Sasso**[2], **V. Umanità**[3]

[1] Dipartimento di Matematica dell'Università di Pavia, via Ferrata 1, 27100 Pavia, Italy.
E-mail: raffaella.carbone@unipv.it
[2] Dipartimento di Matematica dell'Università di Genova, via Dodecaneso 35, 16146 Genova, Italy.
E-mail: sasso@dima.unige.it
[3] Dipartimento di Matematica dell'Università di Genova, via Dodecaneso 35, 16146 Genova, Italy.
E-mail: umanita@dima.unige.it

Abstract: We discuss and extend the Blanchard and Olkiewicz's definition of decoherence for Quantum Markov Semigroups on $\mathcal{B}(\mathsf{h})$ (we will not ask complete positivity). In particular, in the case $\mathsf{h} = \mathbb{C}^2$, we completely characterize the decomposition $\mathcal{B}(\mathsf{h}) = \mathcal{M}_1 \oplus \mathcal{M}_2$ of $\mathcal{B}(\mathsf{h})$ in the sum of a decoherence-free part $\mathcal{M}_1$ and of a space $\mathcal{M}_2$ on which the semigroup vanishes with time.

1. Introduction

When a quantum system associated with a complex separable Hilbert space h interacts with an environment, its evolution becomes irreversible and it is approximately described by a weak*-continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of bounded, completely positive, normal and identity preserving operators acting on the algebra $\mathcal{B}(\mathsf{h})$. $\mathcal{T}$ is called Quantum Markov Semigroup (QMS), and it corresponds to the Heisenberg picture in the sense that, given an observable x, $\mathcal{T}_t(x)$ describes the evolution of x at time t. The dynamics of the states of the system are instead given by the predual semigroup $\mathcal{T}_*$.

In this regime, a loss of phase coherence, as a consequence of the coupling with the environment, has been established (see [21]) and experimentally verified. This means that, for a large time, the evolution of quantum states becomes essentially described by density matrices which are diagonal or block-diagonal with respect to a suitable basis; such a basis can be selected either by the interaction with the environment or by a measurement performed on the system. While for some applications (solid state physics, classical limit) the convergence of states to a diagonal form is an essential step to describe the appearance of classical laws, it may be an undesired effect for other instances (quantum optics, quantum computation), since the destruction of quantum coherences does not allow to distinguish pure states from their corresponding mixtures. So, in these applications it is important to minimize the impact of decoherence, for example by selecting sectors in the space of states that undergo unitary evolution and are thus preserved by dissipation (the so called 'decoherence-free' subspaces) [15].

However, decoherence is presently a very popular topic in quantum physics (see for example [5,13,14,21]). In view of this wide interest, the mathematical study of models for decoherence seems definitely inadequate and the related literature shows this lack. As far as we know, up to now a satisfactory theory of decoherence has not been developed and the few proposed axiomatizations are inequivalent. For example, in [19,20], a simple definition is given under the assumption that $\mathcal{T}$ is ergodic, forcing the decoherence to be a property only of the ergodic state. The definition introduced by Blanchard and Olkiewicz ([2]) seems to fit better with the physical understanding of decoherence. Under the hypothesis of the existence of a faithful, normal, invariant state, it is based on the decomposition of the algebra $\mathcal{B}(\mathsf{h})$ into two parts: a first one, $\mathcal{A}(\mathcal{P})$, the algebra generated by the set $\mathcal{P}$ of preserved projections, which represents the decoherence-free part of the system and coincides with the biggest algebra on which the evolution is unitary, and a second one given by the observables which after some time are not-detectable by measurements. In particular, this approach unifies both the physical problems raised above. In [2,18], the mentioned decomposition is proved for a QMS imposing other conditions on the semigroup which unfortunately seem to be rather restrictive in some cases of physical interest (see for instance Example 1).

In this paper, we would like to go deeper in the study of decoherence, starting from Blanchard and Olkiewicz's definition (BO), because of its considerably good description of certain physical phenomena. In particular, we think there are at least two interesting (and naturally emerging) matters to tackle.
The first problem is a better understanding of the mentioned definition. In Section 2 we analyze a new characterization of the algebra $\mathcal{A}(\mathcal{P})$ that allows us to express the decoherence definition in a new form: this definition is equivalent to the BO's one in the case of QMS, but we introduce it also for semigroups which are not necessarily completely positive (CP) and which do not need to possess a faithful invariant state. We underline that this does not mean that we are not aware that the CP case is in general the most popular in applications, but is simply due to the fact that the complete positivity is not necessary in the definition and does not seem to play any role for the decoherence of the semigroups on $M_2(\mathbb{C})$ that we treat subsequently.
The second problem is finding sufficient and/or necessary conditions in order to identify the decoherence decomposition of $\mathcal{B}(\mathsf{h})$ when a semigroup $\mathcal{T}$ is given. This seems to be a very hard problem to deal with: indeed, the conditions required by BO are sometimes difficult to verify and anyway surely not necessary (as we will see later). This question is quite far from being solved for a QMS acting on a general space. A natural step is starting to face this question in the easiest possible space $\mathcal{B}(\mathsf{h})$, that is the 2×2 matrices. This approach has different motivations: the "easy" context obviously offers many technical advantages but the problem is not trivial, anyway; moreover $\mathsf{h} = \mathbb{C}^2$ is not only a toy space, since it is useful for some interesting physical model (the Wigner-Weisskopf atom, for instance, that we study in Example 1); finally, a complete solution of the problem (i.e. to say exactly when decoherence occurs and to describe the associated decomposition) for a particular choice of h can give interesting suggestions about

the possible extensions of the results, for example about which conditions could be necessary for decoherence.

We conclude the introduction with a few more precise words about the content and the organization of the paper. As we already told, Section 2 is devoted to a discussion of the decoherence definition in a general context (for QMS and also dropping CP for simple positivity). We will express it through a decomposition of the algebra $\mathcal{B}(\mathsf{h})$, $\mathcal{B}(\mathsf{h}) = \mathcal{M}_1 \oplus \mathcal{M}_2$, where the algebra $\mathcal{M}_1$ represents the decoherence-free part of the system and $\mathcal{M}_2$ is contained in the space of not-detectable observables, i.e. the space where the semigroup vanishes with time. The decomposition is proved to be equivalent to the one introduced by BO for QMS taking $\mathcal{M}_1 = \mathcal{A}(\mathcal{P})$; for a more general family of semigroups, we will have anyway $\mathcal{M}_1 \subseteq \mathcal{A}(\mathcal{P})$. Despite this is only a short part of the work, we think it is a central point for our considerations about decoherence.
Section 3 introduces the mathematical objects necessary in order to treat the positivity and identity preserving semigroups on $\mathcal{B}(\mathsf{h}) = M_2(\mathbb{C})$, which will be treated in the rest of the work. Sections 4 and 5 respectively determine explicitly the algebra $\mathcal{A}(\mathcal{P})$ (containing $\mathcal{M}_1$) and the space of not-detectable observables (containing $\mathcal{M}_2$). Finally, in Section 6, we are able to state the main result, Theorem 3, showing when decoherence occurs, and identifying the desired decomposition $M_2(\mathbb{C}) = \mathcal{M}_1 \oplus \mathcal{M}_2$ through the spectrum of the infinitesimal generator, say $\mathcal{L}$, of the semigroup. The section also contains the application to the two-level atom and some conclusive considerations. It is important to remark that, when one thinks about decoherence, it is soon evident that this feature has a strongly relation with the spectrum of the infinitesimal generator (which will essentially be a 4×4 matrix, when $\mathsf{h} = \mathbb{C}^2$). The detailed development of this intuition will bring to an easy and explicit expression of the conditions for decoherence and of the exact decomposition.

2. The environment induces decoherence on the system

In this section we recall and discuss the definition of decoherence introduced by Blanchard and Olkiewicz (BO). We investigate some aspects of this definition in order to understand if some of their requirements are superfluous and in order to highlight which are the most significant features of the spaces involved in the decoherence decomposition (see Corollary 1 and Theorem 1). As a consequence, we shall arrive to write the decoherence definition in a different form.

Let $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ be a Quantum Markov Semigroup (QMS) on the algebra $\mathcal{B}(\mathsf{h})$ of all linear and bounded operators on a complex Hilbert space h, and assume there exists a faithful, normal and invariant state φ. Denote by $\mathcal{A}(\mathcal{P})$ the von Neumann algebra generated by the set $\mathcal{P}$ of all projections P in $\mathcal{B}(\mathsf{h})$ such that $\mathcal{T}_t(P)$ remains a projection for any $t \geq 0$. Blanchard and Olkiewicz say that the *environment induces decoherence* on the system described by $\mathcal{T}$, if there exists a Banach $*$-invariant subspace $\mathcal{M}_2$ in $\mathcal{B}(\mathsf{h})$ such that:

(BO1) $\mathcal{B}(\mathsf{h}) = \mathcal{A}(\mathcal{P}) \oplus \mathcal{M}_2$ with $\mathcal{M}_2 \neq 0$, $\mathcal{A}(\mathcal{P})$ and $\mathcal{M}_2$ $\mathcal{T}_t$-invariant for all $t \geq 0$;
(BO2) for any projection $P \in \mathcal{A}(\mathcal{P})$ and $t \geq 0$ there exists a projection $Q \in \mathcal{A}(\mathcal{P})$ with $\mathcal{T}_t(Q) = P$;
(BO3) $\lim_{t\to\infty} \mathrm{tr}(\rho \mathcal{T}_t(b)) = 0$ for all $b \in \mathcal{M}_2$ and ρ positive trace class operator.

First of all, we intend to better understand the structure of the algebra $\mathcal{A}(\mathcal{P})$ generated by preserved projections.

In [2, Theorem 4] is proved that the complete positivity of the semigroup assures that every restriction $\mathcal{T}_{t|\mathcal{A}(\mathcal{P})}$ is a $*$-homomorphism; moreover, when property (BO2) holds, the faithfulness of φ gives the bijectivity of $\mathcal{T}_{t|\mathcal{A}(\mathcal{P})}$. These properties and the following Proposition suggest us to study the relationship between the algebra $\mathcal{A}(\mathcal{P})$ and the space

$$\mathcal{N}(\mathcal{T}) := \{ \; a \in \mathcal{B}(\mathsf{h}) \; : \; \mathcal{T}_t(a^*a) = \mathcal{T}_t(a^*)\mathcal{T}_t(a), \;\; \mathcal{T}_t(aa^*) = \mathcal{T}_t(a)\mathcal{T}_t(a^*) \;\; \forall\, t \geq 0 \; \} \tag{1}$$

(see [7–9, 11]).

Proposition 1. *If $\mathcal{T}$ is a QMS on $\mathcal{B}(\mathsf{h})$, then $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ on which the action of any $\mathcal{T}_t$ is a $*$-homomorphism. In particular $\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_t$-invariant.*

Proof. $\mathcal{N}(\mathcal{T})$ is clearly self-adjoint. Now define

$$D_t(x, y) := \mathcal{T}_t(x^*y) - \mathcal{T}_t(x^*)\mathcal{T}_t(y), \qquad x, y \in \mathcal{B}(\mathsf{h}),\, t \geq 0.$$

Since each $\mathcal{T}_t$ is completely positive, D_t is a positive sesquilinear form such that

$$D_t(x, x) = 0 \Leftrightarrow D_t(x, y) = 0 \;\; \forall\, y \in \mathcal{B}(\mathsf{h}). \tag{2}$$

Therefore, since $D_t(z, z) = 0 \;\; \forall z \in \mathcal{N}(\mathcal{T})$ and $t \geq 0$, we have that

$$\mathcal{T}_t(x^*y) = \mathcal{T}_t(x^*)\mathcal{T}_t(y) \;\; \forall\, t \geq 0, \;\; \text{if either } x \text{ or } y \text{ belongs to } \mathcal{N}(\mathcal{T}). \tag{3}$$

This easily implies that $\mathcal{N}(\mathcal{T})$ is a vector space. Moreover, if $a, b \in \mathcal{N}(\mathcal{T})$,

$$\begin{aligned}\mathcal{T}_t((ab)^*(ab)) &= \mathcal{T}_t(b^*a^*ab) = \mathcal{T}_t(b^*)\mathcal{T}_t(a^*ab) = \mathcal{T}_t(b^*)\mathcal{T}_t(a^*)\mathcal{T}_t(ab)\\ &= \mathcal{T}_t(b^*a^*)\mathcal{T}_t(ab) = \mathcal{T}_t((ab)^*)\mathcal{T}_t(ab)\end{aligned}$$

for all $t \geq 0$, and

$$\begin{aligned}\mathcal{T}_t((ab)(ab)^*) &= \mathcal{T}_t((b^*a^*)^*(b^*a^*)) = \mathcal{T}_t((b^*a^*)^*)\mathcal{T}_t(b^*a^*)\\ &= \mathcal{T}_t(ab)\mathcal{T}_t((ab)^*)\end{aligned}$$

for all $t \geq 0$, so that $ab \in \mathcal{N}(\mathcal{T})$. Thus, $\mathcal{N}(\mathcal{T})$ is a $*$-subalgebra of $\mathcal{B}(\mathsf{h})$ and it is clearly $\mathcal{T}_t$-invariant for all $t \geq 0$.
Finally, we prove that $\mathcal{N}(\mathcal{T})$ is weak* closed in $\mathcal{B}(\mathsf{h})$. Since (3) shows that $a \in \mathcal{N}(\mathcal{T}) \Leftrightarrow \mathcal{T}_t(b^*a) = \mathcal{T}_t(b^*)\mathcal{T}_t(a) \;\; \forall\, b \in \mathcal{B}(\mathsf{h})$ and $t \geq 0$, if we define

$$\varphi_y : \mathcal{B}(\mathsf{h}) \ni x \mapsto \mathcal{T}_t(y^*x) - \mathcal{T}_t(y^*)\mathcal{T}_t(x) \in \mathcal{B}(\mathsf{h})$$

$\forall\, y \in \mathcal{B}(\mathsf{h})$, we have $\mathcal{N}(\mathcal{T}) = \cap_{y \in \mathcal{B}(\mathsf{h})} \varphi_y^{-1}(\{0\})$, and as a consequence $\mathcal{N}(\mathcal{T})$ is weak* closed, for $\mathcal{T}_t$ is normal for every $t \geq 0$ and the map $x \mapsto yx$ is weakly*-continuous. Therefore, $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ on which the action of any $\mathcal{T}_t$ is a $*$-homomorphism.

Corollary 1. *If $\mathcal{T}$ is a QMS on $\mathcal{B}(\mathsf{h})$, then $\mathcal{A}(\mathcal{P}) = \mathcal{N}(\mathcal{T})$. In particular $\mathcal{A}(\mathcal{P})$ is always $\mathcal{T}_t$-invariant.*

Proof. The algebra $\mathcal{A}(\mathcal{P})$ is contained in the algebra $\mathcal{N}(\mathcal{T})$ since the generating set $\mathcal{P}$ is contained in $\mathcal{N}(\mathcal{T})$. Indeed, take $P \in \mathcal{P}$, then $\mathcal{T}_t(P)$ is a projection and we deduce $\mathcal{T}_t(P^*P) = \mathcal{T}_t(PP^*) = \mathcal{T}_t(P) = (\mathcal{T}_t(P))^2 = (\mathcal{T}_t(P))(\mathcal{T}_t(P^*))$.
Conversely, if $P \in \mathcal{N}(\mathcal{T})$ is a projection, then $\mathcal{T}_t(P)$ remains a projection too by definition of $\mathcal{N}(\mathcal{T})$. So the projections of $\mathcal{N}(\mathcal{T})$ are contained in $\mathcal{P}$ and, since a von Neumann algebra is generated by its projections, $\mathcal{N}(\mathcal{T})$ is contained in $\mathcal{A}(\mathcal{P})$.
This proves that $\mathcal{A}(\mathcal{P}) = \mathcal{N}(\mathcal{T})$ and so it is $\mathcal{T}_t$-invariant.

As a consequence of the previous results, we obtain the following useful characterization of $\mathcal{A}(\mathcal{P})$.

Theorem 1. *Let $\mathcal{T}$ be a QMS on $\mathcal{B}(\mathsf{h})$ possessing a faithful, normal and invariant state. Then $\mathcal{A}(\mathcal{P})$ satisfies property (BO2) if and only if $\mathcal{A}(\mathcal{P})$ is the biggest von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ on which the action of every $\mathcal{T}_t$ is a $*$-automorphism.*

Proof. If $\mathcal{A}(\mathcal{P})$ verifies (BO2), we have already noted that the restriction $\mathcal{T}_{t|\mathcal{A}(\mathcal{P})}$ is a $*$-automorphism. Since $\mathcal{A}(\mathcal{P}) = \mathcal{N}(\mathcal{T})$, it is clear that it is the biggest von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ having this property.

Conversely, given $t \geq 0$ and $P \in \mathcal{A}(\mathcal{P})$ a projection, since the action of $\mathcal{T}_t$ on $\mathcal{A}(\mathcal{P})$ is surjective, we can write $P = \mathcal{T}_t(x)$ for some $x \in \mathcal{A}(\mathcal{P}) = \mathcal{N}(\mathcal{T})$. Thus, the relations

$$\mathcal{T}_t(x) = P = P^* = \mathcal{T}_t(x^*), \qquad \mathcal{T}_t(x) = P = P^2 = \mathcal{T}_t(x)^2 = \mathcal{T}_t(x^2)$$

and the injectivity of $\mathcal{T}_t$ imply $x = x^* = x^2$, i.e. also x is a projection. We can then conclude that $\mathcal{A}(\mathcal{P})$ verifies (BO2).

Remark 1. Under the hypothesis of the previous Theorem, a sufficient condition to guarantee (BO2) is the uniform continuity of $\mathcal{T}$. Indeed, in this case, by Corollary 2.1 in [9] we have $\mathcal{T}_t(x) = U_t^* x U_t$ for all $x \in \mathcal{N}(\mathcal{T})$, with U_t a unitary operator. Therefore, the restriction $\mathcal{T}_{t|\mathcal{N}(\mathcal{T})}$ is a $*$-automorphism.

As a consequence of Theorem 1, the definition by Blanchard and Olkiewicz can be rewritten in an equivalent way by substituting $\mathcal{A}(\mathcal{P})$ with the biggest von Neumann algebra on which each $\mathcal{T}_t$ is a $*$-automorphism. In this way, $\mathcal{B}(\mathsf{h})$ is decomposed in two parts: a first one undergoing unitary evolution (the decoherence free-part), and a second one vanishing for large time. This approach would seem to well describe the physical setting which appears when decoherence takes place.

Now, we aim at extending the definition of decoherence to more general open quantum systems whose dynamics are described by semigroups of positive identity preserving operators $\mathcal{T}_t$, which are not necessarily completely positive. Moreover, we do not require the existence of a faithful, normal and invariant state. We think that a natural generalization of the notion of decoherence could be the following

Definition 1. *Let $\mathcal{T}$ be a weakly*-continuous semigroup of positive, identity preserving and normal bounded maps on $\mathcal{B}(\mathsf{h})$. We say that the* environment induces decoherence *(EID) on the system described by $\mathcal{T}$, if there are two Banach $*$-invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathcal{B}(\mathsf{h})$ such that:*

(i) $\mathcal{B}(\mathsf{h}) = \mathcal{M}_1 \oplus \mathcal{M}_2$ *with* $\mathcal{M}_2 \neq 0$, $\mathcal{M}_1$ *and* $\mathcal{M}_2$ $\mathcal{T}_t$*-invariant for all* $t \geq 0$*;*
(ii) $\mathcal{M}_1$ *is a maximal von Neumann subalgebra of* $\mathcal{B}(\mathsf{h})$ *on which every* $\mathcal{T}_t$ *acts as a* $*$*-automorphism;*
(iii) $\lim_{t\to\infty} \mathrm{tr}(\rho\mathcal{T}_t(b)) = 0$ *for all* $b \in \mathcal{M}_2$ *and* ρ *positive trace class operator.*

If environment induces decoherence, then we call $\mathcal{M}_1$ the *algebra of effective observables*, in the sense that, for t big enough, we can only observe the observables in $\mathcal{A}(\mathcal{P})$, since

$$\lim_{t\to\infty} \mathrm{tr}(\rho\mathcal{T}_t(a)) = \lim_{t\to\infty} \mathrm{tr}(\rho\mathcal{T}_t(a_1))$$

for every self-adjoint $a = a_1 + a_2$ in $\mathcal{B}(\mathsf{h})$ with $a_1 \in \mathcal{A}(\mathcal{P})$ and $a_2 \in \mathcal{M}_2$. It corresponds to the decoherence-free part of the system, the dynamics being unitary on it.
We also call $\mathcal{M}_2$ the *space of decoherent observables.*

Remark 2. Observe that, if $\mathcal{A}$ is a von Neumann algebra such that $\mathcal{T}_{t|\mathcal{A}}$ is a $*$-automorphism, then $\mathcal{A} \subseteq \mathcal{A}(\mathcal{P})$. Indeed if $P \in \mathcal{A}$ is a projection, then $\mathcal{T}_t(P)$ remains a projection too for all $t \geq 0$, and then $P \in \mathcal{P} \subseteq \mathcal{A}(\mathcal{P})$. Since $\mathcal{A}$ is generated by its projections, the claim follows.
Therefore, if environment induces decoherence, $\mathcal{M}_1$ is contained in $\mathcal{A}(\mathcal{P})$. This fact is an important starting point in order to identify $\mathcal{M}_1$ since the procedure to construct $\mathcal{A}(\mathcal{P})$ is in general clear.

Notice that our definition is equivalent to the one by Blanchard and Olkiewicz for QMS, while, if the semigroup is not completely positive, the two definitions are different since, in this case, the action of $\mathcal{T}_t$ on $\mathcal{A}(\mathcal{P})$ is not necessarily a $*$-automorphism (see Theorem 2). Then we obtain two different notions of decoherence according to the decomposition of $\mathcal{B}(\mathsf{h})$ we require, i.e. if we consider the algebra of effective observables either the algebra $\mathcal{A}(\mathcal{P})$ satisfying (BO1), (BO2) or the biggest (or maximal) von Neumann algebra $\mathcal{M}_1$ on which the semigroup acts in a unitary way. We also remark that, due to the loss of complete positivity, the characterization $\mathcal{A}(\mathcal{P}) = \mathcal{N}(\mathcal{T})$ is not always true, since the set $\mathcal{N}(\mathcal{T})$ do not need to be an algebra (see Remark 5).

In the following we restrict our attention to a positive identity preserving semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ on $M_2(\mathbb{C})$, and our aim is to describe subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ for which the decomposition $M_2(\mathbb{C}) = \mathcal{M}_1 \oplus \mathcal{M}_2$ introduced in the previous definition is possible. Since h is finite dimensional, condition *(iii)* in Definition 1 is obviously equivalent to the norm convergence.

3. Positive identity preserving semigroups on $M_2(\mathbb{C})$

In this section we introduce some notations and the essential mathematical objects we need in order to describe positive identity preserving semigroups on $M_2(\mathbb{C})$.
We consider the Pauli's matrices

$$\sigma_0 = \frac{1}{2}\mathbb{1}_{M_2(\mathbb{C})}, \quad \sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as an orthogonal basis of $M_2(\mathbb{C})$, with respect to the scalar product $\langle x, y\rangle_{M_2(\mathbb{C})} = \operatorname{tr}(x^* y)$. They satisfy

$$\sigma_k^2 = \sigma_0^2, \qquad \sigma_k\sigma_j = -\sigma_j\sigma_k, \qquad \sigma_k\sigma_j = i\sigma_l\sigma_0$$

for $(k, j, l) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. An arbitrary element x of $M_2(\mathbb{C})$ can be written as $x = w_0\sigma_0 + w \cdot \sigma$, where $w_0 \in \mathbb{C}$, $w = (w_1, w_2, w_3)$ in $\mathbb{C}^3$ and $w \cdot \sigma := \sum_{k=1}^3 w_k\sigma_k$.
We recall some trivial properties, which will be useful for computations.

(a) If $x = w_0\sigma_0 + w \cdot \sigma$ and $y = z_0\sigma_0 + z \cdot \sigma$, $(w, z \in \mathbb{C}^3,\ w_0, z_0 \in \mathbb{C})$ then

$$2xy = (w_0z_0 + \langle \overline{w}, z\rangle)\sigma_0 + (w_0z + z_0w + iw \wedge z)\cdot\sigma, \quad 2\operatorname{tr}(x^*y) = \bar{w}_0z_0 + \langle w, z\rangle,$$

(here $\langle\cdot,\cdot\rangle$ is the inner product and $\wedge$ the exterior product in $\mathbb{C}^3$).

(b) $x = w_0\sigma_0 + w \cdot \sigma$ is *positive* if and only if $w_0 \in \mathbb{R}$, $w \in \mathbb{R}^3$ and $\|w\| \le w_0$.

Let $\mathcal{T} = (\mathcal{T}_t)_{t\ge0}$ be a norm continuous semigroup of linear operators on $M_2(\mathbb{C})$ and let $\mathcal{L}$ be its generator. Clearly $\mathcal{L}$, as linear operator on $M_2(\mathbb{C})$, can be represented as a 4×4 matrix in the basis $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$. Moreover, it is easy to see that:

i) $\mathcal{T}$ is adjoint preserving (i.e. $\mathcal{T}_t(x^*) = \mathcal{T}_t(x)^*$ for all $x \in M_2(\mathbb{C})$) if and only if the matrix representing $\mathcal{L}$ is real,

ii) $\mathcal{T}$ is identity preserving if and only if the first column of the matrix representing $\mathcal{L}$ is zero.

Therefore the matrix representing $\mathcal{L}$ can be written as follows

$$\mathcal{L} = \begin{pmatrix} 0 & b^T \\ o & A \end{pmatrix} \tag{4}$$

where b is a column vector in $\mathbb{R}^3$, b^T denotes its transpose, o is the zero vector in $\mathbb{R}^3$ and A is in $M_3(\mathbb{R})$. Then

$$\mathcal{T}_t = \exp(t\mathcal{L}) = \sum_n \frac{t^n\mathcal{L}^n}{n!} = \sum_n \frac{t^n}{n!}\begin{pmatrix} 0 & b^TA^{n-1} \\ o & A^n \end{pmatrix} = \begin{pmatrix} 1 & \int_0^t b^Te^{sA}ds \\ o & e^{tA} \end{pmatrix},$$

or, equivalently, for $w_0 \in \mathbb{C}$, $w \in \mathbb{C}^3$,

$$\mathcal{T}_t(w_0\sigma_0 + w\cdot\sigma) = \left(w_0 + \int_0^t \langle b, e^{sA}w\rangle ds\right)\sigma_0 + \left(e^{tA}w\right)\cdot\sigma. \tag{5}$$

An identity preserving semigroup $\mathcal{T}$ is called *positive* if it preserves the cone of positive operators in $M_2(\mathbb{C})$ (i.e. $\mathcal{T}_t(x) \ge 0$ for all $x \ge 0$ and all t). This is equivalent to have

$$\langle v, (A + A^*)v\rangle \le 2\|v\|\langle b, v\rangle, \tag{6}$$

for all v in $\mathbb{R}^3$ (see [6, Proposition 2.1].
Remember that, if $\mathcal{T}$ is positive, it is necessarily adjoint preserving.

Proposition 2. *If $\mathcal{T}$ is a positive and identity preserving semigroup, then the following facts hold:*

a) $A + A^* \le 0$, and $\Re\mathrm{e}\lambda \le 0$ for any λ eigenvalue of A;

b) $\mathrm{Ker(A + A^*)} \subseteq \mathrm{b}^\perp$.

Proof. 1. For all $v \in \mathbb{R}^3$, use (6) for v and $-v$, and get

$$\langle v, (A + A^*)v \rangle \leq -2\|v\| |\langle b, v \rangle| \leq 0. \tag{7}$$

Now, let λ be an eigenvalue of A with eigenvector v, $\|v\| = 1$. Then

$$2\Re e \lambda = 2\Re e \langle v, Av \rangle = \langle v, (A + A^*)v \rangle \leq 0$$

gives the thesis.
2. If $v \in \mathrm{Ker(A + A^*)}$, then (7) becomes $0 \leq -2\|v\| |\langle b, v \rangle| \leq 0$. So $\langle b, v \rangle = 0$.

Notice that, by (6), if $b = 0$, the condition $A + A^* \leq 0$ is also sufficient for the adjoint and identity preserving semigroup $\mathcal{T}$ to be positive.

Remark 3. If $A + A^* \leq 0$, then $\mathrm{Ker(A)} \subseteq \mathrm{Ker(A + A^*)}$.
Indeed, if $w \in \mathrm{Ker(A)}$ and $A + A^* \leq 0$, then $\|(-(A + A^*))^{1/2} w\|^2 = \langle w, (A + A^*)w \rangle = 2\Re e \langle w, Aw \rangle = 0$, so w is in $\mathrm{Ker(A + A^*)}$.

From now on we shall consider positive and identity preserving semigroups, so $\mathcal{T}$ will be generated by a matrix $\mathcal{L}$ of the form (4) with b in $\mathbb{R}^3$ and A a 3×3 real matrix verifying condition (6). We shall also denote by K the $\mathrm{Ker(A + A^*)}$.

4. $\mathcal{M}_1$ and the algebra $\mathcal{A}(\mathcal{P})$ of "preserved" projections

As we already explained, we want to investigate decoherence of semigroups defined on the space $M_2(\mathbb{C})$. We choose to search for a decomposition of the domain, as in Definition 1, starting from the description of a possible space $\mathcal{M}_1$. Since, when $\mathcal{T}$ is a QMS, this algebra coincides with $\mathcal{A}(\mathcal{P})$ and, in general, $\mathcal{M}_1 \subseteq \mathcal{A}(\mathcal{P})$ (see Remark 2), it is quite natural that we first aim to find a characterization of the projections "preserved" by the semigroup, that is we want to describe the set

$$\mathcal{P} := \{ P \text{ proj} : \; \mathcal{T}_t(P) \;\; \text{proj} \; \forall \, t \geq 0 \; \}$$

(Proposition 3).
Then we shall study the algebra generated by $\mathcal{P}$ (Proposition 4) and we will investigate if the restriction of $\mathcal{T}_t$ to this algebra is a $*$-automorphism (Theorem 2).

A **projection** is a self-adjoint operator P such that $P^2 = P$. Since we are working in $M_2(\mathbb{C})$, it is easy to see that $P = w_0 \sigma_0 + w \cdot \sigma$ is a projection iff

$$P = \mathbb{1} \text{ or } P = \sigma_0 + w \cdot \sigma, \text{ with } w \in \mathbb{R}^3, \|w\| = 1.$$

The identity is always preserved by the semigroups we consider; for other projections we will use the following Lemma and Proposition in order to understand if their images still are projections.

Lemma 1. *If $w \in \mathbb{C}^3$, then*

a) $\|e^{tA} w\| = \|w\|$ *for all t iff $e^{tA} w \in K$ for all t iff $w \in \mathcal{S} := \cap_{n=0}^{2} A^{-n}(K) = \cap_{n \geq 0} A^{-n}(K)$.*

b) $\int_0^t \langle b, e^{sA}w\rangle ds = 0$ *for all* t *iff* $A^n w \perp b$ *for* $n \geq 0$ *iff* $A^n w \perp b$ *for* $n = 0, 1, 2$.

Proof. *a)* We can compute

$$\frac{d}{dt}\|e^{tA}w\|^2 = \langle e^{tA}w, (A + A^*)e^{tA}w\rangle,$$

and, remembering that $A + A^* \leq 0$ by the positivity of the semigroup,

$$\|e^{tA}w\|^2 = \|w\|^2 + \int_0^t \langle e^{sA}w, (A+A^*)e^{sA}w\rangle\, ds = \|w\|^2 - \int_0^t \|(-(A+A^*))^{1/2}e^{sA}w\|^2\, ds.$$

So $\|e^{tA}w\|$ is constant iff $e^{tA}w \in K$ for all $t \geq 0$. By using the series expansion of $e^{tA}w$, it is obvious that $e^{tA}w \in K$ for all $t \geq 0$ iff $A^n w \in K$ for all $n \geq 0$. We still have to prove that it is sufficient to have $A^n w \in K$ for all $n = 0, 1, 2$.
Indeed, if $w, Aw, A^2 w \in K$, then
- if they are l.i., then they generate all the space $\mathbb{C}^3$ and so $K = \mathbb{C}^3$ obviously contains all the $A^n w$'s;
- if $Aw = \alpha w$ for some $\alpha \in \mathbb{C}$, then $A^n w = \alpha^n w \in K$ for all n;
- if the two previous conditions are not true, then $A^2 w = \alpha w + \beta A w$ for some α and β, and so $A^{n+2}w = A^n(A^2 w) = A^n(\alpha w + \beta A w) = \alpha A^n w + \beta A^{n+1} w$ and this is in K by induction arguments.

b) $\int_0^t \langle b, e^{sA}w\rangle ds = 0$ for all t iff $\int_0^t e^{sA}w\, ds \perp b$ for all t by linearity. Using the series expansion of the function $t \mapsto \int_0^t e^{sA}w\, ds$, we have that this integral is orthogonal to b iff all the derivatives (in zero) are, so iff $A^n w \perp b$ for $n \geq 0$.
In order to prove that it is sufficient to have $A^n w \perp b$ for all $n = 0, 1, 2$, one proceeds as in the proof of *a*).

Now, we may characterize the preserved projections.

Proposition 3. *Let* $w \in \mathbb{C}^3$*, the following statements are equivalent:*

a) $\mathcal{T}_t(\sigma_0 + w \cdot \sigma)$ *is a projection for all* t*;*
b) $w \in \mathbb{R}^3$, $\int_0^t \langle b, e^{sA}w\rangle ds = 0$ *and* $\|e^{tA}w\| = 1$ *for all* t*;*
c) $w \in \mathbb{R}^3$, $\|w\| = 1$ *and* $w \in \mathcal{S}$.

In particular, we have that $\mathcal{P} = \{2\sigma_0\} \cup \{\sigma_0 + w \cdot \sigma \,:\, w \in \mathcal{S} \cap \mathbb{R}^3, \|w\| = 1\}$.

Proof. Call $P = \sigma_0 + w \cdot \sigma$. Then

$$\mathcal{T}_t(P) = \left(1 + \int_0^t \langle b, e^{sA}w\rangle ds\right)\sigma_0 + \left(e^{tA}w\right) \cdot \sigma$$

is a projection for all t iff, for all t,

$$\begin{gathered} \int_0^t \langle b, e^{sA}w\rangle ds = 1 \text{ and } e^{tA}w = 0 \\ \text{or} \\ w \in \mathbb{R}^3,\ \int_0^t \langle b, e^{sA}w\rangle ds = 0 \text{ and } \|e^{tA}w\| = 1. \end{gathered} \tag{8}$$

But, $e^{tA}w = 0$ for some t iff $w = 0$ and this would also imply $\int_0^t \langle b, e^{sA}w\rangle ds = 0$ for all t, so the first condition in (8) cannot hold. Then $\mathcal{T}_t(P)$ is a projection for all t iff the second condition in (8) is true for all t. So the equivalence of a) and b) is proved.
The equivalence between the b) and c) is an immediate consequence of the previous Lemma, just remember that K is orthogonal to b by Proposition 2.
This proves also that $\mathcal{P} = \{2\sigma_0\} \cup \{\sigma_0 + w \cdot \sigma \,:\, w \in \mathcal{S} \cap \mathbb{R}^3, \|w\| = 1\}$.

Remark 4. The space $\mathcal{S}$, introduced in Lemma 1, has a real basis, so the dimension of $\mathcal{S}$ as a complex subspace of $\mathbb{C}^3$ is equal to the dimension of $\mathcal{S} \cap \mathbb{R}^3$ as a real subspace of $\mathbb{R}^3$.
Indeed, when we search for the vectors of K, we have to solve the linear system $(A + A^*)w = 0$. Since $A + A^*$ is real (because A is real) and symmetric, it has a diagonal form in $\mathbb{R}$, so there exists a real basis for K. This will give a real basis also for $\mathcal{S}$.
Namely, if $\mathcal{S} = K$ or $\mathcal{S} = \{0\}$, it is obviously true. The only case which is excluded is when K has dimension 2 and $\mathcal{S}$ has dimension 1. Then, let $w^1, w^2 \in \mathbb{R}^3$ be a basis for K and take a vector $v \in \mathbb{C}^3$ generating $\mathcal{S}$. Since $v, Av, A^2v \in K$, we have, for some $\alpha_i, \beta_i, \gamma_i$ in $\mathbb{C}$,

$$v = \alpha_1 w^1 + \alpha_2 w^2, \qquad Av = \beta_1 w^1 + \beta_2 w^2, \qquad A^2 v = \gamma_1 w^1 + \gamma_2 w^2$$

and so, passing to the real and imaginary parts

$$\Re e v = \Re e \alpha_1 w^1 + \Re e \alpha_2 w^2, \quad A(\Re e v) = \Re e \beta_1 w^1 + \Re e \beta_2 w^2, \quad A^2(\Re e v) = \Re e \gamma_1 w^1 + \Re e \gamma_2 w^2$$

and similarly for $\Im m v$. So $\Re e v$ and $\Im m v$ are vectors in $\mathcal{S}$ and at least one of them is non-null, so it can be taken as a basis for $\mathcal{S}$.

With the previous results, we can finally conclude about the algebra $\mathcal{A}(\mathcal{P})$, generated by $\mathcal{P}$. In the following, we denote by d the dimension of $\mathcal{S}$.

Proposition 4. $\mathcal{A}(\mathcal{P}) = \operatorname{span} \mathcal{P}$ *for* $d \neq 2$ *and in particular*

a) if $d = 0$*, then* $\mathcal{A}(\mathcal{P}) = \mathbb{C}\mathbb{1}$*;*
b) if $d = 1$*, then* $\mathcal{A}(\mathcal{P}) = \operatorname{span}\{\mathbb{1}, w \cdot \sigma\}$*, with* $\operatorname{span}\{w\} = \cap_{n=0}^2 A^{-n}(K)$*;*
c) if $d = 2$ *or* $d = 3$*, then* $\mathcal{A}(\mathcal{P}) = M_2(\mathbb{C})$*.*

Moreover, $\mathcal{A}(\mathcal{P})$ *is a* $\mathcal{T}_t$*-invariant von Neumann algebra.*

Proof. By Proposition 3, statements a) and b) are clear. We have only to prove the thesis for $d = 2$. In this case we can consider $w_1 \perp w_2$ such that $\mathcal{S} = \operatorname{span}\{w_1, w_2\}$. Then $\mathbb{1}$, $(w_1 \cdot \sigma)$ and $(w_2 \cdot \sigma)$ surely belong to $\mathcal{A}(\mathcal{P})$ and, since $\mathcal{A}(\mathcal{P})$ is an algebra, it also contains

$$(w_1 \cdot \sigma)(w_2 \cdot \sigma) = \langle w_1, w_2\rangle \sigma_0 + i(w_1 \wedge w_2) \cdot \sigma = i(w_1 \wedge w_2) \cdot \sigma,$$

which is linearly independent of $\operatorname{span}\{\mathbb{1}, w_1 \cdot \sigma, w_2 \cdot \sigma\}$; it follows that $\mathcal{A}(\mathcal{P}) = M_2(\mathbb{C})$.
We still have to prove that $\mathcal{A}(\mathcal{P})$ is a $\mathcal{T}_t$-invariant von Neumann algebra. Since h is finite-dimensional, we only have to show that it is closed under the involution $*$ and $\mathcal{T}_t$-invariant. It is immediate when $d = 0$ or when $d = 2, 3$. When $d = 1$,

just notice that $\mathcal{A}(\mathcal{P}) = \text{span}\{\mathbb{1}, w \cdot \sigma\}$, with $\text{span}\{w\} = \cap_{n=0}^{2} A^{-n}(K)$ and w can be choosen in $\mathbb{R}^3$ by Remark 4. So if $x = \alpha\mathbb{1} + \beta w \cdot \sigma$ is a generic element of $\mathcal{A}(\mathcal{P})$, then $x^* = \bar{\alpha}\mathbb{1} + \bar{\beta} w \cdot \sigma$ also belongs to $\mathcal{A}(\mathcal{P})$. Moreover, by Lemma 1 and the first part of this proposition, it is easy to see that $\mathcal{T}_t(\mathcal{A}(\mathcal{P})) \subseteq \mathcal{A}(\mathcal{P})$.

Now we want to verify when we can choose the algebra $\mathcal{A}(\mathcal{P})$ as $\mathcal{M}_1$ defined in Definition 1. We just know that $\mathcal{A}(\mathcal{P})$ is $\mathcal{T}_t$-invariant: thus we have to investigate when $\mathcal{T}_{t|_{\mathcal{A}(\mathcal{P})}}$ is a $*$-automorphism and if $\mathcal{A}(\mathcal{P})$ is maximal wrt such property.

Theorem 2. *a) $\mathcal{T}_t$ is a $*$-automorphism on $\mathcal{A}(\mathcal{P})$ if and only if $d \neq 2$.*
b) If $d \neq 2$, $\mathcal{A}(\mathcal{P})$ is the biggest algebra on which $\mathcal{T}_t$ is a $*$-automorphism.
c) If $d = 2$, every $\mathcal{A}_w := \text{span}\{\mathbb{1}, w \cdot \sigma\}$, with $w \in \mathcal{S}$, is a maximal algebra on which $\mathcal{T}_t$ is a $*$-automorphism.

Proof. *a*) Remember that $\mathcal{T}_t$ is a $*$-automorphism on $\mathcal{A}(\mathcal{P})$ if it is a bijective operator on $\mathcal{A}(\mathcal{P})$ verifying $\mathcal{T}_t(x^*y) = \mathcal{T}_t(x)^*\mathcal{T}_t(y)$ for all x and y in $\mathcal{A}(\mathcal{P})$.
Here $\mathcal{T}_t = e^{t\mathcal{L}}$ is always bijective since it is an invertible operator in $M_2(\mathbb{C})$ and $\mathcal{T}_t(\mathcal{A}(\mathcal{P})) \subseteq \mathcal{A}(\mathcal{P})$.
Now, consider $x = w_0\sigma_0 + w \cdot \sigma$ and $y = z_0\sigma_0 + z \cdot \sigma$, ($w, z \in \mathbb{C}^3$, $w_0, z_0 \in \mathbb{C}$) and compute

$$\begin{aligned}
\mathcal{T}_t(x^*y) &= \frac{1}{2}\mathcal{T}_t((\overline{w}_0 z_0 + \langle w, z\rangle)\sigma_0 + (\overline{w}_0 z + z_0\overline{w} + i\overline{w} \wedge z) \cdot \sigma) \\
&= \frac{1}{2}\left(\overline{w}_0 z_0 + \langle w, z\rangle + \int_0^t \langle b, e^{sA}(\overline{w}_0 z + z_0 \overline{w} + i\overline{w} \wedge z)\rangle ds\right)\sigma_0 \\
&+ \frac{1}{2}\left(e^{tA}(\overline{w}_0 z + z_0\overline{w} + i\overline{w} \wedge z)\right) \cdot \sigma \\
\mathcal{T}_t(x)^*\mathcal{T}_t(y) &= \left(\overline{(w_0 + \int_0^t \langle b, e^{sA} w\rangle ds)}\sigma_0 + \overline{e^{tA} w} \cdot \sigma\right)\left((z_0 + \int_0^t \langle b, e^{sA} z\rangle ds)\sigma_0 + (e^{tA} z) \cdot \sigma\right) \\
&= \frac{1}{2}\left[(\overline{w}_0 + \int_0^t \langle b, e^{sA}\overline{w}\rangle ds)(z_0 + \int_0^t \langle b, e^{sA} z\rangle ds) + \langle e^{tA}\overline{w}, e^{tA} z\rangle\right]\sigma_0 \\
&+ \frac{1}{2}\left\{(\overline{w}_0 + \int_0^t \langle b, e^{sA}\overline{w}\rangle ds)e^{tA} z + (z_0 + \int_0^t \langle b, e^{sA} z\rangle ds)e^{tA}\overline{w} + i(e^{tA}\overline{w}) \wedge (e^{tA} z)\right\} \cdot \sigma.
\end{aligned}$$

Therefore, we have $\mathcal{T}_t(x^*y) = \mathcal{T}_t(x)^*\mathcal{T}_t(y)$ if and only if

$$\begin{aligned}
\langle w, z\rangle + i\int_0^t \langle b, e^{sA}(\overline{w} \wedge z)\rangle ds = (&\int_0^t \langle b, e^{sA}\overline{w}\rangle ds)(\int_0^t \langle b, e^{sA} z\rangle ds) \\
&+ \langle e^{tA}\overline{w}, e^{tA} z\rangle
\end{aligned} \tag{9}$$

$$\begin{aligned}
ie^{tA}(\overline{w} \wedge z) = (&\int_0^t \langle b, e^{sA}\overline{w}\rangle ds)e^{tA} z + (\int_0^t \langle b, e^{sA} z\rangle ds)e^{tA}\overline{w} \\
&+ i(e^{tA}\overline{w}) \wedge (e^{tA} z)
\end{aligned} \tag{10}$$

If $d = 0$, then $\mathcal{A}(\mathcal{P}) = \mathbb{C}\mathbb{1}$ and $\mathcal{T}_t$ is a $*$-automorphism.

If $d = 1$, $\mathcal{S} = \mathrm{span}\{w\}$, then $\mathcal{A}(\mathcal{P}) = \mathrm{span}\{\mathbb{1}, w \cdot \sigma\}$. Then we have to check $\mathcal{T}_t(x^*y) = \mathcal{T}_t(x)^*\mathcal{T}_t(y)$ for x and y in $\mathrm{span}\{\mathbb{1}, w \cdot \sigma\}$. This is easily verified remembering that

$$\int_0^t \langle b, e^{sA}w\rangle ds = 0 \text{ and } \|e^{tA}w\| = \|w\|.$$

If $d = 2$ or $d = 3$, then $\mathcal{A}(\mathcal{P}) = M_2(\mathbb{C})$, so equalities (9) and (10) have to hold for all vectors $w, z \in \mathbb{C}^3$.
Fix $t \geq 0$ and suppose that $\mathcal{T}_t$ is a $*$-automorphism. Using (10) for $z = \overline{w}$, we obtain

$$(\int_0^t \langle b, e^{sA}\overline{w}\rangle ds)e^{tA}\overline{w} = 0 \qquad \forall\, w$$

so, since $e^{tA}\overline{w} = 0$ only when $\overline{w} = 0$, we have $\int_0^t \langle b, e^{sA}\overline{w}\rangle ds = 0$ for any w. This clearly implies $\int_0^t \langle b, e^{sA}w\rangle ds = 0$ for any $w \in \mathbb{C}^3$.

Therefore

$$(9) + (10) \Rightarrow \begin{cases} \int_0^t \langle b, e^{sA}w\rangle ds = 0 \\ \langle w, z\rangle = \langle e^{tA}w, e^{tA}z\rangle \\ e^{tA}(\overline{w} \wedge z) = (e^{tA}\overline{w}) \wedge (e^{tA}z) \end{cases} \qquad \text{for all } w, z \in \mathbb{C}^3$$

$$\Rightarrow \begin{cases} \int_0^t \langle b, e^{sA}w\rangle ds = 0 \quad \text{for all } w \in \mathbb{C}^3 \quad (1') \\ e^{tA} = e^{-tA^*} \quad (2') \\ e^{tA} = det(e^{tA})e^{-tA^*}, \end{cases}$$

since $(e^{tA}\overline{w}) \wedge (e^{tA}z) = det(e^{tA})e^{-tA^*}(\overline{w} \wedge z)$. In particular $det(e^{tA}) = 1$ holds. Deriving $(1')$ and $(2')$ in $t = 0$, we get $\langle b, w\rangle = 0$ for all $w \in \mathbb{C}^3$ and $A + A^* = 0$ respectively, so that $b = 0$ and $K = \mathbb{R}^3$. This clearly implies $d = 3$.

Conversely, if $d = 3$, then $\mathcal{S} = \mathbb{C}^3$, so that $A + A^* = 0$ and $b = 0$. It follows that (9) and (10) hold, and then $\mathcal{T}_t$ is a $*$-homomorphism.

b) The statement for $d \neq 2$ follows by *a*) and Remark 2.

c) Suppose now $d = 2$. $\mathcal{T}_t$ acts as a $*$-automorphism on $\mathcal{A}_w$ as a consequence of the previous proposition. It remains to observe that $\mathcal{A}_w$ is maximal. But if we consider an algebra $\mathcal{B} \supsetneq \mathcal{A}_w$, then there exists a vector $w' \in \mathbb{C}^3$ linearly independent with w, such that $w' \cdot \sigma \in \mathcal{B}$. Therefore, since the product

$$(w' \cdot \sigma)(w \cdot \sigma) = \langle w', w\rangle\sigma_0 + i(w' \wedge w) \cdot \sigma$$

belongs to $\mathcal{B}$ and $w' \wedge w$ is linearly independent with w, w', we get $\mathcal{B} = M_2(\mathbb{C})$. This is a contradiction because for $d = 2$ the action of $\mathcal{T}_t$ is not a $*$-automorphism on the whole space $\mathcal{A}(\mathcal{P}) = M_2(\mathbb{C})$.

Remark 5. By previous result, if $d = 2$, there does not exist the biggest algebra on which the action of the semigroup is unitary. So $\mathcal{T}_t$ is not able to be completely positive (see Proposition 1).
Moreover, always for $d = 2$, the set $\mathcal{N}(\mathcal{T})$ cannot be an algebra.
Indeed, $\mathcal{P}$ coincides with the set of the projections of $\mathcal{N}(\mathcal{T})$ (see the proof of Corollary 1). So, if $\mathcal{N}(\mathcal{T})$ was an algebra, we would get $\mathcal{A}(\mathcal{P}) \subseteq \mathcal{N}(\mathcal{T})$ and then $\mathcal{N}(\mathcal{T}) = M_2(\mathbb{C})$, by Proposition 4. This would also imply $d = 3$ (see Proposition 3).

5. Characterization of not-detectable observables

In the previous section, we have found the possible candidates to the algebra of effective observables $\mathcal{M}_1$; now we start analyzing the spectral proprieties of A, in order to characterize the space

$$\mathcal{M}_2' := \{x \in M_2(\mathbb{C}) \,:\, \lim_{t\to\infty} \mathcal{T}_t(x) = 0\}$$

which contains $\mathcal{M}_2$ (Proposition 6). We call $\mathcal{M}_2'$ the *s*pace of not-detectable observables.

Let us consider the matrix A, and call Λ the set of its eigenvalues. We denote by N_λ the principal space and by V_λ the eigenspace associated with an eigenvalue λ in Λ, so that

$$\mathbb{C}^3 = \oplus_{\lambda\in\Lambda} N_\lambda. \tag{11}$$

In particular, by the positivity of $-(A+A^*)$, if $\Re e\lambda = 0$, then $N_\lambda = V_\lambda$. Indeed, if $z \in N_\lambda$ is an element of the Jordan basis, but it is not an eigenvector, then there exists a non-null eigenvector v, such that $Az = v + \lambda z$. Therefore,

$$\begin{aligned}
\langle \alpha v+\beta z, (A+A^*)(\alpha v+\beta z)\rangle &= 2\Re e\langle \alpha v+\beta z, A(\alpha v+\beta z)\rangle \\
&= 2\Re e\langle \alpha v+\beta z, \alpha\lambda v+\beta v+\beta\lambda z)\rangle \\
&= 2\Re e\{\bar\alpha(\alpha\lambda+\beta)\|v\|^2 + \lambda|\beta|^2\|z\|^2 \\
&+ \bar\beta(\alpha\lambda+\beta)\langle z,v\rangle + \bar\alpha\beta\lambda\langle v,z\rangle\} \\
(\text{ since } \Re e\lambda = 0) &= 2\Re e\bar\alpha\beta\|v\|^2 + 2\Re e\{\lambda(\bar\beta\alpha\langle z,v\rangle + \bar\alpha\beta\langle v,z\rangle)\} \\
&+ |\beta|^2 2\Re e\langle z,v\rangle = 2\Re e\bar\alpha\beta\|v\|^2 + |\beta|^2 2\Re e\langle z,v\rangle.
\end{aligned}$$

So

$$\langle \alpha v+\beta z, (A+A^*)(\alpha v+\beta z)\rangle \le 0 \quad\Leftrightarrow\quad \Re e\{\bar\alpha\beta\} \le -\frac{|\beta|^2}{\|v\|^2}\Re e\{\langle z,v\rangle\}.$$

This cannot obviously be true for all α and β in $\mathbb{C}$.
Thus, we may read the decomposition (11) as

$$\mathbb{C}^3 = \left(\oplus_{\Re e\lambda=0} V_\lambda\right) \oplus \left(\oplus_{\Re e\lambda<0} N_\lambda\right),$$

and the first part of the following result is proved.

Proposition 5. *We have*

$$\mathbb{C}^3 = J_0 \oplus J_-$$

where $J_0 = \oplus_{\Re e\lambda=0} V_\lambda$ is the vector space generated by the eigevectors of A associated with $\lambda \in \Lambda$ with $\Re e\lambda = 0$, and $J_- = \oplus_{\Re e\lambda<0} N_\lambda$ is the vector space generated by the principal spaces associated with $\lambda \in \Lambda$ with $\Re e\lambda < 0$. The spaces J_0 and J_- are preserved by the operator A.
Moreover $\|e^{tA}z\| \to_{t\to\infty} 0$ whenever $z \in J_-$ and $J_0 = \mathcal{S}$.

Proof. The spaces J_0 and J_- are obviously preserved by the operator A since the principal spaces are.
Now we prove that $\|e^{tA}z\| \to 0$ whenever $z \in J_-$. We consider only when $dim N_\lambda = 3$, since the case $dim N_\lambda = 2$ is very similar and the case $dim N_\lambda = 1$

is immediate. In this context, we have $A = UJU^{-1}$, with U an invertible matrix and J the Jordan form of A,

$$J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix};$$

so that $e^{tA} = Ue^{tJ}U^{-1}$ with

$$e^{tJ} = \begin{pmatrix} e^{t\lambda} & te^{t\lambda} & \frac{1}{2}t^2e^{t\lambda} \\ 0 & e^{t\lambda} & te^{t\lambda} \\ 0 & 0 & e^{t\lambda} \end{pmatrix}.$$

Let $\{f_1, f_2, f_3\}$ be a Jordan basis of N_λ, then, for all $z \in J_-$, $z = \sum_{i=1}^3 z_i f_i$, we have

$$\begin{aligned} \|e^{tA}z\| &= e^{t\Re e\lambda}\|z_1 f_1 + (tz_1 + z_2)f_2 + (\frac{t^2}{2}z_1 + tz_2 + z_3)f_3\| \\ &\leq e^{t\Re e\lambda}\left(\|z\| + t\|z_1 f_2 + z_2 f_3\| + \frac{t^2}{2}\|z_1 f_3\|\right) \end{aligned}$$

and this proves our claim for $\Re e\lambda < 0$.
Finally, we show that $J_0 = \cap_n A^{-n}(K)$. Take an eigenvector v of A associated with the eigenvalue λ with $\Re e\lambda = 0$. Then

$$\|(-(A + A^*))^{1/2}v\|^2 = -\langle v, (A + A^*)v\rangle = -2\Re e\langle v, Av\rangle = -2\Re e\lambda\|v\|^2 = 0.$$

So $v \in K$, $A^n v = \lambda^n v \in K$ for all n, and then $v \in \cap_n A^{-n}(K)$. This proves $J_0 \subseteq \cap_n A^{-n}(K)$.
Conversely, given $v \in \cap_n A^{-n}(K)$, we can write $v = z_1 + z_2$ for some unique $z_1 \in J_0$ and $z_2 \in J_-$, and consequently $z_2 = v - z_1 \in A^{-n}(K)$ as difference of two elements of the same space. This implies $z_2 = 0$, since $\|e^{tA}z_2\|$ is constant wrt t for $z_2 \in A^{-n}(K)$ (by Lemma 1) and $\|e^{tA}z_2\| \to 0$ since $z_2 \in J_-$. Therefore, $v = z_1$ belongs to J_0.

We can now characterize the space $\mathcal{M}_2'$.

Proposition 6. *The space of not-detectable observables is*

$$\mathcal{M}_2' = \{\, \langle b, \tilde{A}^{-1}w\rangle\sigma_0 + w \cdot \sigma : w \in J_- \}$$

with $\tilde{A} := A_{|J_-}$. *In particular, this is a Banach $*$-subspace of $M_2(\mathbb{C})$ which is $\mathcal{T}_t$-invariant.*

Proof. If $x = w_0\sigma_0 + w \cdot \sigma \in M_2(\mathbb{C})$, then

$$\mathcal{T}_t(x) = \left(w_0 + \int_0^t \langle b, e^{sA}w\rangle ds\right)\sigma_0 + \left(e^{tA}w\right)\cdot\sigma \to 0 \Leftrightarrow \begin{cases} \int_0^t \langle b, e^{sA}w\rangle ds \to -w_0 \\ e^{tA}w \to 0. \end{cases}$$

Since $w = z_1 + z_2$, with $z_1 \in J_0$ and $z_2 \in J_-$, by Proposition 5, we have

$$\lim_t e^{tA}w = \lim_t e^{tA}z_1,$$

so $e^{tA}w \to 0$ iff $z_1 = 0$ by Proposition 3, since $\|e^{tA}z\| = \|z\|$ for all $z \in \mathcal{S}$. This is equivalent to have $w \in J_-$. Now consider the restriction of the operator A, $\tilde{A} := A_{|J_-}$. This is invertible since its eigenvalues are surely different from 0; moreover $\tilde{A}^n = (A^n)_{|J_-}$, since J_- is preserved by A, so one also has $e^{s\tilde{A}} = (e^{sA})_{|J_-}$ for all s. Then, for $w \in J_-$,

$$\int_0^t \langle b, e^{sA}w\rangle ds = \langle b, \int_0^t e^{s\tilde{A}} ds\, w\rangle = \langle b, \tilde{A}^{-1}(e^{t\tilde{A}} - 1)w\rangle \to \langle b, -\tilde{A}^{-1}w\rangle. \quad (12)$$

Therefore, we can conclude that $x \in \mathcal{M}_2'$ if and only if $w \in J_-$ and $w_0 = \langle b, \tilde{A}^{-1}w\rangle$.
We prove now the last part of the statement. It is clear that $\mathcal{M}_2'$ is a Banach $*$-subspace. Moreover, given $w \in J_-$, by (12) we have

$$\begin{aligned}\mathcal{T}_t(\langle b, \tilde{A}^{-1}w\rangle\sigma_0 + w \cdot \sigma) &= \left(\langle b, \tilde{A}^{-1}w\rangle + \int_0^t \langle b, e^{sA}w\rangle ds\right)\sigma_0 + (e^{tA}w)\cdot\sigma \\ &= \langle b, \tilde{A}^{-1}e^{t\tilde{A}}w\rangle\sigma_0 + (e^{tA}w)\cdot\sigma \\ &= \langle b, \tilde{A}^{-1}e^{tA}w\rangle\sigma_0 + (e^{tA}w)\cdot\sigma\end{aligned}$$

with $e^{tA}w \in J_-$ since J_- is preserved by A thanks to Proposition 5. This implies that $\mathcal{M}_2'$ is $\mathcal{T}_t$-invariant.

6. Decoherence for positive semigroups

In this section we can finally state the main result (Theorem 3), which tells exactly when the EID holds, and explicitly determine subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$. The results collected in previous sections now allow us to prove this theorem very easily, so the section is mainly devoted to the description of a noteworthy application and to some final considerations, for instance about new comparisons with the definition of EID given by Blanchard and Olkiewicz, or about possible developments of this research.

The previous results assure that the following decomposition of the algebra $M_2(\mathbb{C})$ always holds.

Proposition 7. *If $\mathcal{T}$ is a positive (identity preserving) semigroup then*

$$M_2(\mathbb{C}) = \mathrm{span}\mathcal{P} \oplus \mathcal{M}_2'.$$

Proof. By Propositions 3 and 6 we have $dim\,(\mathrm{span}\mathcal{P}) = dim\,\mathcal{S} + 1 = d + 1$, while $dim\mathcal{M}_2' = dimJ_- = 3 - d$. So, since $dim(\mathrm{span}\mathcal{P}) + dim\mathcal{M}_2 = 4$, it is enough to prove that the vectors in span$\mathcal{P}$ and in $\mathcal{M}_2'$ are linearly independent. Therefore, let $w_1 \in \mathcal{S}$ and $w_2 \in J_-$ and assume that

$$w_0\sigma_0 + w_1 \cdot \sigma = \alpha(\langle b, \tilde{A}^{-1}w_2\rangle\sigma_0 + w_2 \cdot \sigma),$$

for some $\alpha \in \mathbb{C} \setminus \{0\}$. Then we have that $w_0 = \alpha\langle b, \tilde{A}^{-1}w_2\rangle$, and $w_1 = \alpha w_2$. The second equality implies $w_1 = w_2 = 0$ for $\mathcal{S} \cap J_- = \{0\}$ and $\alpha \neq 0$. So by the first condition, we get $w_0 = 0$ too. This concludes the proof.

As a consequence, we obtain the characterization for the EID.

Theorem 3. *The EID holds if and only if either $d = 0$ or $d = 1$. Moreover, in these cases, $\mathcal{M}_1$ and $\mathcal{M}_2$ are uniquely determined and we have $\mathcal{M}_1 = \mathcal{A}(\mathcal{P})$ and $\mathcal{M}_2 = \{ \langle b, \tilde{A}^{-1}w\rangle\sigma_0 + w \cdot \sigma : w \in J_- \}$.*

If $d = 3$, the semigroup acts in a unitary way on the whole space $M_2(\mathbb{C})$ and $\mathcal{M}_2' = \{0\}$

Proof. Assume $d \neq 2$. Then span$\mathcal{P} = \mathcal{A}(\mathcal{P})$ by Proposition 4, and $\mathcal{A}(\mathcal{P})$ is the biggest von Neumann subalgebra of $M_2(\mathbb{C})$ on which every $\mathcal{T}_t$ acts as a $*$-automorphism thanks to Theorem 2. Therefore, set $\mathcal{M}_1 = \mathcal{A}(\mathcal{P})$ and $\mathcal{M}_2 = \mathcal{M}_2' := \{ \langle b, \tilde{A}^{-1}w\rangle\sigma_0 + w\cdot\sigma : w \in J_- \}$. For $d = 0, 1$, the decomposition $M_2(\mathbb{C}) = \mathcal{M}_1 \oplus \mathcal{M}_2$ satisfies the desired properties of the EID definition; for $d = 3$, we have $J_- = \{0\}$ and then $\mathcal{M}_2' = \{0\}$, so the EID cannot hold since we would be forced to choose $\mathcal{M}_2 = \{0\}$. Note that $\mathcal{M}_1$ and $\mathcal{M}_2$ are uniquely determined thanks to Proposition 7, since $\mathcal{M}_1 \subseteq \mathcal{A}(\mathcal{P}) = \text{span}\mathcal{P}$ and $\mathcal{M}_2 \subseteq \mathcal{M}_2'$.
For the converse, suppose that the EID holds for $d = 2$. Since in this case $dim\,\mathcal{M}_2' = 3 - d = 1$ and $\mathcal{M}_2 \neq \{0\}$, we necessarily have $\mathcal{M}_2 = \mathcal{M}_2'$, so that $dim\,\mathcal{M}_1 = 3$. But this contradicts Theorem 2.

The last statement is a consequence of Theorem 2 and Proposition 6.

Example 1. (Two level atom). We now analyse when the EID holds for semigroups arising in the study of the irreversible evolutions of the squeezed Wigner-Weisskopf or two level atom (see for example [10], [12]).

Let us define $\sigma_+ = \sigma_1 + i\sigma_2$ and $\sigma_- = \sigma_1 - i\sigma_2 = \sigma_+^*$, and let $\mathcal{L}_\zeta$ ($\zeta \in \mathbb{C}, \omega \in \mathbb{R}$) be the map on $M_2(\mathbb{C})$

$$\mathcal{L}_0(x) = -\frac{\lambda^2}{2}(\sigma_-\sigma_+x - 2\sigma_-x\sigma_+ + x\sigma_-\sigma_+) - \frac{\mu^2}{2}(\sigma_+\sigma_-x - 2\sigma_+x\sigma_- + x\sigma_+\sigma_-),$$

$$\mathcal{L}_{\zeta,\omega}(x) = \mathcal{L}_0(x) - \zeta\sigma_+x\sigma_+ - \overline{\zeta}\sigma_-x\sigma_- + i\omega[\sigma_3, x].$$

The positive constants λ and μ are jumps rates from the lower to the upper and upper to the lower energy level. The complex parameter ζ (see [10]) represents *squeeze* and satisfies $|\zeta| \leq \lambda\mu$. When the squeeze parameter ζ vanishes we find the generator of the usual master equation for the Wigner-Weisskopf atom. One can verify that $\mathcal{L}_{\zeta,\omega}$ is the generator of a QMS $\mathcal{T}$. In the Pauli's basis, it is represented as

$$\mathcal{L}_{\zeta,\omega} = \begin{pmatrix} 0 & 0 & 0 & \lambda^2 - \mu^2 \\ 0 & -\frac{\lambda^2+\mu^2}{2} - \Re e\zeta & \Im m\zeta + \omega & 0 \\ 0 & \Im m\zeta - \omega & \Re e\zeta - \frac{\lambda^2+\mu^2}{2} & 0 \\ 0 & 0 & 0 & -\lambda^2 - \mu^2 \end{pmatrix}.$$

By Theorem 3, in order to exploit whenever EID holds, we have to find the dimension d of $\mathcal{S}$, or equivalently of J_0, i.e. the dimensions of eingenspaces corresponding to eingenvalues of

$$A = \begin{pmatrix} -\frac{\lambda^2+\mu^2}{2} - \Re e\zeta & \Im m\zeta + \omega & 0 \\ \Im m\zeta - \omega & \Re e\zeta - \frac{\lambda^2+\mu^2}{2} & 0 \\ 0 & 0 & -\lambda^2 - \mu^2 \end{pmatrix},$$

having real part equal to 0.

It is straightforward to verify that the eingeinvalues are

$$\left.\begin{array}{l}\lambda_1=-\frac{\lambda^2+\mu^2}{2}+\sqrt{|\zeta|^2-\omega^2}\\ \lambda_2=-\frac{\lambda^2+\mu^2}{2}-\sqrt{|\zeta|^2-\omega^2}\\ \lambda_3=-\lambda^2-\mu^2\end{array}\right. \qquad \text{if } |\zeta|^2-\omega^2\geq 0$$

and

$$\left.\begin{array}{l}\lambda_1=-\frac{\lambda^2+\mu^2}{2}+i\sqrt{-|\zeta|^2+\omega^2}\\ \lambda_2=-\frac{\lambda^2+\mu^2}{2}-i\sqrt{|\zeta|^2+\omega^2}\\ \lambda_3=-\lambda^2-\mu^2\end{array}\right. \qquad \text{if } |\zeta|^2-\omega^2< 0.$$

Excluding the trivial case $\lambda=\mu=0$, we have

$$d=1 \ \Leftrightarrow\ \begin{cases}|\zeta|^2-\omega^2>0\\ \frac{\lambda^2+\mu^2}{2}=\sqrt{|\zeta|^2-\omega^2}\end{cases},$$

and in this case,

$$J_0=\mathrm{span}\{z\}\quad \text{with } z=(\Im m\zeta+\omega,\sqrt{|\zeta|^2-\omega^2}+\Re e\zeta,0),$$
$$J_-=\mathrm{span}\{(\Im m\zeta+\omega,\Re e\zeta-\sqrt{|\zeta|^2-\omega^2},0),(0,0,1)\},$$

so that EID holds with

$$\mathcal{M}_1=\mathrm{span}\{\mathbb{1},z\cdot\sigma\}=\mathrm{span}\{\mathbb{1},(\Im m\zeta+\omega)\sigma_1+(\sqrt{|\zeta|^2-\omega^2}+\Re e\zeta)\sigma_2\}$$
$$\mathcal{M}_2=\mathrm{span}\{(\Im m\zeta+\omega)\sigma_1+(\Re e\zeta-\sqrt{|\zeta|^2-\omega^2})\sigma_2,\frac{\mu^2-\lambda^2}{2\sqrt{|\zeta|^2-\omega^2}}\sigma_0+\sigma_3\}.$$

In all other cases we get $d=0$, and so EID still holds with $\mathcal{M}_1=\mathbb{C}\mathbb{1}$ and

$$\mathcal{M}_2=\left\{\frac{\mu^2-\lambda^2}{\lambda^2+\mu^2}z_3\sigma_0+z\cdot\sigma\ :\ z\in\mathbb{C}^3\right\}.$$

Thus we always obtain that EID holds, even if the sufficient conditions required in [2, Theorem 10] and in [4, Section 2.2] by Blanchard and Olkiewicz are not satisfied at least when $|\zeta|<\frac{\lambda^2+\mu^2}{2}$. Indeed in this case, an easy computation shows that the trace is not an invariant state, while

$$\rho=\sigma_0+\frac{\lambda^2-\mu^2}{\lambda^2+\mu^2}\sigma_3$$

is a faithful invariant state (see [6]) such that the modular automorphism associated with it does not commute with the generator $\mathcal{L}_{\zeta,\omega}$.

A conclusive parallel between our EID definition and BO's one.

1. What happens if we used the BO's EID, simply dropping the complete positivity of the semigroup (but preserving positivity)? Theorem 3 shows in particular that the result would be essentially the same.

For $d=0,1$, the algebra of effective observables which we found coincides with the one described by Blanchard and Olkiewicz in their definition of decoherence:

indeed, in these cases, $\mathcal{M}_1$ is really the algebra $\mathcal{A}(\mathcal{P})$ of preserved projections and it clearly satisfies property (BO2). Therefore, since the space $\mathcal{M}_2$ is characterized in the same way, for $d = 0, 1$ our decomposition $M_2(\mathbb{C}) = \mathcal{M}_1 \oplus \mathcal{M}_2$ is exactly the one desired by Blanchard and Olkiewicz. If $d = 3$, since $\mathcal{M}_2'$ is zero, the environment does not induce decoherence neither in our meaning nor in the Blanchard and Olkiewicz's one. This signifies that, for $d \neq 2$, the complete positivity of the maps $\mathcal{T}_t$ does not play a basic role in the decoherence analysis of the system $\mathbb{C}^2$.

2. Always looking forward a comparison with the results by Blanchard and Olkiewicz, maybe it is worth underlining that, when the EID holds, the algebra $\mathcal{M}_1$ is the image of a conditional expectation.

Proposition 8. *When the EID holds, there exists a unique conditional expectation $\mathcal{E} : M_2(\mathbb{C}) \to \mathcal{M}_1$ onto $\mathcal{M}_1$ and it is given in the following way:*

1. *if $d = 0$, then $\mathcal{E}(x) = \mathrm{tr}(x)/2$,*
2. *if $d = 1$, then*

$$\mathcal{E}(z_0\sigma_0 + z \cdot \sigma) = z_0\sigma_0 + \langle w, z\rangle\, w \cdot \sigma \qquad \forall\, z_0 \in \mathbb{C},\ z \in \mathbb{C}^3, \tag{13}$$

where $w \in \mathbb{R}^3$ is the real norm-one vector generating $\mathcal{S}$.

Proof. The result is clear for $d = 0$ (and also the expression of the conditional expectation is immediate).
When $d = 1$, we recall that $\mathcal{M}_1 = \mathrm{span}\{\mathbb{1}, w \cdot \sigma\}$, with $\mathcal{S} = \mathrm{span}\{w\}$. Assume there exists a conditional expectation $\mathcal{E}$ onto $\mathcal{M}_1$, i.e. a linear norm one projection onto $\mathcal{M}_1$. Given $x = z_0\sigma_0 + z \cdot \sigma$ an arbitrary element in $M_2(\mathbb{C})$, we must have $\mathcal{E}(x) = z_0\sigma_0 + \mu w \cdot \sigma$ for some $\mu \in \mathbb{C}$. Moreover, the equality

$$\mathcal{E}(axb) = a\mathcal{E}(x)b \tag{14}$$

has to hold for all $a, b \in \mathcal{M}_1$; therefore, choosing $a = b = w \cdot \sigma$, the equalities

$$\begin{aligned} 4\mathcal{E}(axb) &= 2\mathcal{E}\left((\langle w, z\rangle\sigma_0 + (z_0 w + iw \wedge z) \cdot \sigma)(w \cdot \sigma)\right) \\ &= \mathcal{E}\left(\langle \overline{z}_0 w - iw \wedge \overline{z}, w\rangle\sigma_0 + \langle w, z\rangle\, w \cdot \sigma\right) \\ &= z_0\sigma_0 + \langle w, z\rangle\, w \cdot \sigma \end{aligned}$$

and

$$\begin{aligned} 4a\mathcal{E}(x)b &= 4(w \cdot \sigma)\left(z_0\sigma_0 + \mu w \cdot \sigma\right)(w \cdot \sigma) \\ &= 2(\mu\sigma_0 + z_0 w \cdot \sigma)(w \cdot \sigma) = z_0\sigma_0 + \mu w \cdot \sigma \\ &= \mathcal{E}(x) \end{aligned}$$

imply $\mathcal{E}(x) = \mathcal{E}(z_0\sigma_0 + z \cdot \sigma) = z_0\sigma_0 + \langle w, z\rangle\, w \cdot \sigma$ by equation (14).
We now want to show that this operator is really a conditional expectation. The property of projection is self-evident. As first step to show $\|\mathcal{E}\| = 1$, we prove that $\mathcal{E}$ is a positive operator. Let $x = z_0\sigma_0 + z \cdot \sigma$ be a positive element in $M_2(\mathbb{C})$. As already mentioned in Section 3, this means $z_0 \in \mathbb{R}$, $z \in \mathbb{R}^3$ and $z_0 \geq \|z\|$. Since $\mathcal{E}(x) = z_0\sigma_0 + \langle w, z\rangle w \cdot \sigma$ and $\|w\| = 1$, $\mathcal{E}(x)$ is positive if and only if $\langle w, z\rangle w$ belongs to $\mathbb{R}^3$ and $z_0 \geq |\langle w, z\rangle|$; these conditions are clearly satisfied because $w, z \in \mathbb{R}^3$ and $|\langle w, z\rangle| \leq \|z\| \leq z_0$. This proves the positivity of $\mathcal{E}$.

Since the algebra $\mathcal{M}_1$ is commutative, it follows that $\mathcal{E}$ is also completely positive ([1]). Therefore, its norm is given by $\|\mathcal{E}(\mathbb{1})\| = \|\mathbb{1}\| = 1$.

3. In [2], Blanchard and Olkiewicz showed that their decomposition of $\mathcal{B}(\mathsf{h})$ holds, for an arbitrary separable Hilbert space h, assuming the invariance of the trace and the existence of a faitfhul subinvariant state; later, in [4], they weakened these hypothesis supposing the existence of an arbitrary normal semifinite faithful and subinvariant weight such that the modular group associated with it commutes with the semigroup (or equivalently, with its generator).
Here instead, considering the simplest non-trivial Hilbert space $\mathsf{h} = \mathbb{C}^2$, we are able to state when the EID decomposition holds and to identify it without additional hypothesis. Obviously, even if we do not need any assumption about invariant states, it is clear that our semigroups have at least an invariant state, by Markov-Kakutani's theorem. This can easily be verified also directly, since a positive trace one operator ρ is invariant for the semigroup $\mathcal{T}$ iff $\mathrm{tr}(\rho\mathcal{L}(x)) = \mathrm{tr}(\rho x)$ for all x, that is iff, by (4), $\rho = \sigma_0 + u \cdot \sigma$ with $A^*u + b = 0$ (see [6]). Such a vector u always exists since we know that $b \in K^\perp \subseteq \mathrm{Ker(A)}^\perp = \mathrm{Im}(A^*)$; moreover it is real, since A^* and b are, and, by the positivity of the semigroup, $-\langle u, b\rangle = \Re\mathrm{e}\langle u, Au\rangle \leq -\|u\||\langle u, b\rangle|$, so $\|u\| \leq 1$; this guarantees that ρ is positive.

Possible generalizations of EID results (not only about what is concerned with our work).

1. A first question we can wonder about is what happens when the EID decomposition does not hold. Could we introduce a different decomposition of the algebra $\mathcal{B}(\mathsf{h})$? For example by introducing a third space, which becomes trivial when EID holds.
So let us now analyze the case when the EID decomposition does not hold in our context, that is $d = 2$: since the maps $\mathcal{T}_t$ are not completely positive (see Remark 5), the Blanchard and Olkiewicz's definition of decoherence is not applicable, but we can however observe that the algebra $\mathcal{A}(\mathcal{P})$ (equal to $M_2(\mathbb{C})$ in this framework) does not satisfy condition (BO2). More precisely, fixed $t > 0$ and given a projection $P = \sigma_0 + w \cdot \sigma$ with $w \in \mathbb{R}^3 \setminus \mathcal{S}$ and $\|w\| = 1$, there exists no projection Q such that $\mathcal{T}_t(Q) = P$.

Indeed, take $Q = \sigma_0 + z \cdot \sigma$ with $z \in \mathbb{R}^3$ and $\|z\| = 1$ satisfying $\mathcal{T}_t(Q) = P$, we have in particular $z = e^{-tA}w$. Therefore, by equation

$$1 = \|w\|^2 = \|e^{tA}z\|^2 = 1 - \int_0^t \|(-(A + A^*))^{1/2}e^{sA}z\|^2 ds$$

we get $(-(A + A^*))^{1/2}e^{sA}z = 0$ for all $s \in [0, t]$, i.e. $e^{(s-t)A}w = e^{sA}z \in K$ for all $s \leq t$. Then, in particular, $\frac{d^n}{ds^n}e^{sA}z\big|_{s=t} = A^n w \in K$ for all n, and this contradicts the assumption $w \notin \mathcal{S}$.

We can show that, if $d = 2$, a third subspace takes part in the decomposition of $M_2(\mathbb{C})$, but the decomposition we find is not unique.

Proposition 9. *Assume $d = 2$. For every non-zero element $\omega \in \mathcal{S}$ the decomposition*

$$M_2(\mathbb{C}) = \mathcal{A}_\omega \oplus \mathcal{A}_{\omega,\perp} \oplus \mathcal{M}_2' \tag{15}$$

holds with

$$\mathcal{A}_\omega := \mathrm{span}\{\mathbb{1}, w\cdot\sigma\}, \qquad \mathcal{A}_{\omega,\perp} := \mathrm{span}\{w^\perp\cdot\sigma\}$$

and $\mathrm{span}\{w\}\oplus\mathrm{span}\{w^\perp\} = \mathcal{S}$.
In particular, $\mathcal{A}_\omega$ *is a maximal von Neumann subalgebra of* $M_2(\mathbb{C})$ *on which the action of each* $\mathcal{T}_t$ *is a* $*$*-automorphism, and* $\mathcal{M}_2'$ *is a* $\mathcal{T}_t$*-invariant Banach* $*$*-subspace such that* $\mathcal{T}_t(x)\to_{t\to\infty} 0$ *for all* $x\in\mathcal{M}_2'$.

Proof. Fix $\omega\in\mathcal{S}$, $\omega\neq 0$. We only have to prove equation (15). But this is a simple consequence of Proposition 7, since $\mathrm{span}\mathcal{P} = \mathrm{span}\{\mathbb{1}, z\cdot\sigma\}$ with $z\in\mathcal{S}$ and $\mathcal{S} = \mathrm{span}\{w\}\oplus\mathrm{span}\{w^\perp\}$.

2. The second problem which we naturally have to face is the generalization of our results to general algebras $\mathcal{B}(\mathsf{h})$, maybe first passing through the case when h has finite dimension.
Indeed, in our analysis of decoherence, we took strong advantage of knowing the explicit representation of the semigroup $\mathcal{T}$ in terms of the Pauli basis. This tool is clearly unsuited to extend the study of decoherence to QMSs on arbitrary Hilbert spaces h, when the semigroup is known only through its generator. In this case, a characterization of the algebra of effective observables (or, equivalently, of the algebra $\mathcal{N}(\mathcal{T})$) in terms of $\mathcal{L}$ becomes necessary. To go in this direction we remark that

$$\sigma(\mathcal{L}) = \{0\}\cup\sigma(A), \tag{16}$$

$$\mathrm{span}\,\mathcal{P} = \mathrm{span}\{x\in M_2(\mathbb{C})\,|\,\mathcal{L}(x)=\lambda x \text{ with } \Re\mathrm{e}\lambda = 0\}. \tag{17}$$

Namely, by (4), we have $\mathcal{L}(v_0\sigma_0 + v\cdot\sigma) = \langle b, v\rangle\sigma_0 + Av\cdot\sigma$ for any $v_0\in\mathbb{C}$ and $v\in\mathbb{C}^3$. In particular, if we fix λ, an element $x = v_0\sigma_0 + v\cdot\sigma\in M_2(\mathbb{C})$ is such that $\mathcal{L}(x)=\lambda x$ iff $\langle b,v\rangle = \lambda v_0$ and $Av=\lambda v$. When we have $\Re\mathrm{e}\lambda = 0$ this is equivalent to ask $Av=\lambda v$ (so $v\in\mathcal{S}\subseteq b^\perp$) and $\lambda v_0 = 0$. This shows (16) and

$$\mathrm{span}\{x\in M_2(\mathbb{C})\,|\,\mathcal{L}(x)=\lambda x \text{ with } \Re\mathrm{e}\lambda=0\} = \mathrm{span}\{\sigma_0, v\cdot\sigma\,|\,Av=\lambda v,\ \Re\mathrm{e}\lambda=0\},$$

and this vector space is $\mathrm{span}\,\mathcal{P}$ due to Propositions 3 and 5.

Note that, if $d\neq 2$, equation (17) completely characterizes the algebra of effective observable $\mathcal{M}_1$, since in this case it exactly coincides with $\mathrm{span}\,\mathcal{P}$ (see Proposition 4). Therefore, the natural generalization of our results to QMSs on arbitrary Hilbert spaces could be to show that $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$ is equal to the algebra generated by the set $\{x\in M_2(\mathbb{C})\,|\,\mathcal{L}(x)=\lambda x \text{ with } \Re\mathrm{e}\lambda=0\}$. Really, it is easy to prove that, if $\mathcal{T}$ is a uniformly continuous semigroup possessing a faithful invariant state, then the vector space generated by $\{x\in M_2(\mathbb{C})\,|\,\mathcal{L}(x) = \lambda x \text{ with } \Re\mathrm{e}\lambda=0\}$ is always an algebra and a subset of $\mathcal{N}(\mathcal{T})$; moreover, at least when h is finite-dimensional, also the opposite inclusion holds.

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