ADJOINTS FOR MULTIPLE CATEGORIES (ON WEAK AND LAX MULTIPLE CATEGORIES, III)

MARCO GRANDIS AND ROBERT PARÉ

Abstract. Continuing our first two papers in this series, we study adjoint functors for infinite-dimensional multiple categories. The general setting is *chiral multiple categories* - a weak, partially lax form with directed interchanges between the weak composition laws.

Introduction

This is the third paper in a series on weak and lax multiple categories, of finite or infinite dimension - an extension of weak double and weak cubical categories.

Our main framework, a *chiral multiple category*, was introduced in the first article [GP8], cited below as Part I; it is a partially lax multiple category with a strict composition $gf = f +_0 g$ in direction 0 (the *transversal direction*), weak compositions $x +_i y$ in all positive (or *geometric*) directions $i \in \mathbb{N} \setminus \{0\}$ and *directed* interchanges for the i- and j-compositions (for $0 < i < j$)

$$\chi_{ij}(x, y, z, u)\colon (x +_i y) +_j (z +_i u) \to (x +_j z) +_i (y +_j u) \qquad (ij\text{-}interchanger). \qquad (1)$$

Part II [GP9] studies multiple limits in this setting. We now investigate multiple adjoints, extending the study of double adjunctions in [GP2] and cubical adjunctions in [G3].

Section 1 is an informal introduction to multiple adjunctions. After a synopsis of weak and chiral multiple categories, we describe a natural colax/lax adjunction $F \dashv G$ between the weak *double* categories of spans and cospans (already studied in [GP2]) and its extension to the corresponding infinite dimensional, weak *multiple* categories (of cubical type); the functor F is constructed with pushouts and is colax, while G is constructed with pullbacks and is lax. Then we derive from this adjunction other instances, between *chiral* multiple categories that are not of cubical type. Other examples are given in 1.7.

In Section 2 we introduce the strict *double* category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories (or *cm-categories*), lax and colax cm-functors and suitable double cells. Comma cm-categories are also considered. Both topics extend notions of weak double categories developed in [GP2].

Section 3 reviews the notions of companions and adjoints *in a double category*, from [GP2]. Then Sections 4 and 5 introduce and study multiple colax/lax adjunctions, as adjoint arrows *in* the double category $\mathbb{C}\mathrm{mc}$.

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Finally, Section 6 deals with the preservation of limits by right adjoints, for cm-categories.

Literature. Strict double and multiple categories were introduced and studied by C. Ehresmann and A.C. Ehresmann [Eh, BE, EE1, EE2, EE3]. Strict cubical categories can be seen as a particular case of multiple categories (as shown in Part I); their links with strict ω-categories are made clear in [BM, ABS]. Weak double categories (or pseudo double categories) were introduced and studied in our series [GP1] - [GP4]; adjunctions and monads in this setting are also studied in [FGK1, FGK2, Ni]; other aspects are developed in [DPR, Fi, Ga, P2, P3]. For weak cubical categories see [G1] - [G3] and [GP5]. The three-dimensional case of lax triple categories covers and combines diverse structures like duoidal categories, Gray categories, Verity double bicategories and monoidal double categories; see [GP6, GP7]. Further information on literature for higher dimensional category theory can be found in the Introduction of Part I.

Notation. We follow the notation of Parts I and II [GP8, GP9]; the reference I.2.3 or II.2.3 points to Subsection 2.3 of Part I or Part II. The symbol $\subset$ denotes weak inclusion. Categories and 2-categories are generally denoted as $\mathbf{A}, \mathbf{B}$...; weak double categories as $\mathbb{A}, \mathbb{B}$...; weak or lax multiple categories as A, B... More specific points of notation are recalled below, in 1.1.

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1. Some basic examples of adjunctions

We begin by recalling examples of adjunctions for *weak double categories*, studied in [GP2], and their extension to (infinite-dimensional) *weak cubical categories*, studied in [G3]. Then we derive from the latter some instances of adjunctions between *chiral multiple* categories that are not of a cubical type.

This section is an informal introduction to such adjunctions, precise definition will be given later.

1.1. Notation. The definitions of weak and chiral multiple categories can be found in Part I, or - briefly reviewed - in Part II, Section 1. Here we only give a sketch of them, while recalling the notation we are using.

The two-valued index α (or β) varies in the set $2 = \{0, 1\}$, also written as $\{-, +\}$.

A *multi-index* $\mathbf{i}$ is a finite subset of $\mathbb{N}$, possibly empty. Writing $\mathbf{i} \subset \mathbb{N}$ it is understood that $\mathbf{i}$ is finite; writing $\mathbf{i} = \{i_1, ..., i_n\}$ it is understood that $\mathbf{i}$ has n *distinct* elements, written in the natural order $i_1 < i_2 < ... < i_n$; the integer $n \geqslant 0$ is called the *dimension* of $\mathbf{i}$. We write:

$$\mathbf{i}j = j\mathbf{i} = \mathbf{i} \cup \{j\} \qquad (\text{for } j \in \mathbb{N} \setminus \mathbf{i}), \qquad\qquad \mathbf{i}|j = \mathbf{i} \setminus \{j\} \qquad (\text{for } j \in \mathbf{i}). \tag{2}$$

For a weak multiple category A, the set of $\mathbf{i}$-cells $A_{\mathbf{i}}$ is written as A_*, A_i, A_{ij} when $\mathbf{i}$ is $\varnothing, \{i\}, \{i, j\}$ respectively. Faces and degeneracies, satisfying the *multiple relations*, are denoted as

$$\partial_j^\alpha \colon X_{\mathbf{i}} \to X_{\mathbf{i}|j}, \qquad\qquad e_j \colon X_{\mathbf{i}|j} \to X_{\mathbf{i}}. \tag{3}$$

The *transversal direction* $i = 0$ is set apart from the positive, or *geometric*, directions. For a *positive multi-index* $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N} \setminus \{0\}$ the *augmented multi-index* $0\mathbf{i} = \{0, i_1, ..., i_n\}$ has dimension $n+1$, but both $\mathbf{i}$ and $0\mathbf{i}$ have *degree* n. An $\mathbf{i}$-cell $x \in A_{\mathbf{i}}$ of A is also called an $\mathbf{i}$-*cube*, while a $0\mathbf{i}$-cell $f \in A_{0\mathbf{i}}$ is viewed as an $\mathbf{i}$-*map* $f \colon x \to_0 y$, where $x = \partial_0^- f$ and $y = \partial_0^+ f$. Composition in direction 0 is categorical (and generally realised by ordinary composition of mappings); it is written as $gf = f +_0 g$, with identities $1_x = \mathrm{id}(x) = e_0(x)$. The *transversal category* $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ consists of the $\mathbf{i}$-cubes and $\mathbf{i}$-maps of A, with transversal composition and identities.

On the other hand, composition of $\mathbf{i}$-cubes and $\mathbf{i}$-maps in a *positive* direction $i \in \mathbf{i}$ (often realised by pullbacks, pushouts, tensor products, etc.) is written in additive notation

$$\begin{array}{ll} x +_i y & (\partial_i^+ x = \partial_i^- y), \\ f +_i g \colon x +_i y \to_0 x' +_i y' & (f \colon x \to_0 x',\, g \colon y \to_0 y',\, \partial_i^+ f = \partial_i^- g). \end{array} \tag{4}$$

These operations are *categorical* and *interchangeable* up to transversally-invertible comparisons (for $0 < i < j$, see I.3.2)

$$\begin{array}{ll} \lambda_i x \colon (e_i \partial_i^- x) +_i x \to_0 x & (\textit{left i-unitor}), \\ \rho_i x \colon x +_i (e_i \partial_i^+ x) \to x & (\textit{right i-unitor}), \\ \kappa_i(x, y, z) \colon x +_i (y +_i z) \to_0 (x +_i y) +_i z & (\textit{i-associator}), \\ \chi_{ij}(x, y, z, u) \colon (x +_i y) +_j (z +_i u) \to_0 (x +_j z) +_i (y +_j u) & (\textit{ij-interchanger}). \end{array} \tag{5}$$

The comparisons are natural with respect to transversal maps; λ_i, ρ_i and κ_i are *special in direction i* (i.e. their i-faces are transversal identities), while χ_{ij} is *special in both directions i, j*; all of them commute with ∂_k^α for $k \neq i$ (or $k \neq i, j$ in the last case). Finally the comparisons must satisfy various conditions of coherence, listed in I.3.3 and I.3.4.

More generally for a *chiral multiple category* A the ij-interchangers χ_{ij} are not assumed to be invertible (see I.3.7).

1.2. Cubical spans and cospans. Weak multiple categories generalise *weak cubical categories* and *weak symmetric cubical categories*; the latter were introduced in [G1] for studying higher cobordism, and give our main examples of weak multiple categories of infinite dimension. We begin by recalling two instances in an informal, incomplete way.

The weak symmetric cubical category $\mathsf{Span}(\mathbf{C})$ *of cubical spans* (or $\omega\mathsf{Span}(\mathbf{C})$) was constructed in [G1] over a category $\mathbf{C}$ with (a fixed choice of) pullbacks. An n-*cube* is a functor $x\colon \vee^n \to \mathbf{C}$, where $\vee$ is the formal-span category

$$
\begin{tikzcd}
0 & u \arrow[l] \arrow[r] & 1 & \vee
\end{tikzcd}
$$

$$
\begin{tikzcd}
(0,0) & (u,0) \arrow[l] \arrow[r] & (1,0) & & \bullet \arrow[r, "1"] \arrow[d, "2"] & {} \\
(0,u) \arrow[u] \arrow[d] & (u,u) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & (1,u) \arrow[u] \arrow[d] & & {} & \\
(0,1) & (u,1) \arrow[l] \arrow[r] & (1,1) & \vee^2. & &
\end{tikzcd}
\tag{6}
$$

(Identities and composites are understood.) An *n-map*, or transversal map of n-cubes, is a natural transformation $f\colon x \to y\colon \vee^n \to \mathbf{C}$ of such functors; these maps form the category $\mathrm{Span}_n(\mathbf{C}) = \mathbf{Cat}(\vee^n, \mathbf{C})$, with composition written as gf and identities $1_x = \mathrm{id}(x)$. There are obvious *geometric faces and degeneracies* (satisfying the cubical relations)

$$
\begin{aligned}
&\partial_i^\alpha\colon \mathrm{Span}_n(\mathbf{C}) \to \mathrm{Span}_{n-1}(\mathbf{C}),\\
&e_i\colon \mathrm{Span}_{n-1}(\mathbf{C}) \to \mathrm{Span}_n(\mathbf{C}) \qquad (i = 1, ..., n;\ \alpha = \pm).
\end{aligned}
\tag{7}
$$

Moreover there are *geometric* composition laws: the *i-concatenation* $x +_i y$ is defined for *i-consecutive* n-cubes ($i = 1, ..., n$; $\partial_i^+ x = \partial_i^- y$), and constructed with pullbacks; it is categorical up to invertible n-maps (*unitors* and *associators*); similarly we have the i-concatenation $f +_i g$ of i-consecutive n-maps. All pairs of composition laws have a strict interchange.

Viewing $\mathsf{Span}(\mathbf{C})$ as a weak multiple category (of cubical type), an n-cube $x\colon \vee^n \to \mathbf{C}$ is viewed as an $\mathbf{i}$-cube, for every positive multi-index $\mathbf{i} = \{i_1, ..., i_n\}$ of dimension $n \geqslant 0$; an n-map is viewed as an $\mathbf{i}$-map.

The 2-dimensional and 3-dimensional truncations of $\mathsf{Span}(\mathbf{C})$ are written as:

$$\mathbb{S}\mathrm{pan}(\mathbf{C}) = \mathbf{2}\mathsf{Span}(\mathbf{C}), \qquad\qquad \mathbf{3}\mathsf{Span}(\mathbf{C}). \tag{8}$$

The weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$ was studied in our series [GP1] - [GP4]: its horizontal and vertical arrows are ordinary arrows and spans of $\mathbf{C}$, respectively, while a double cell is a morphism of spans. The 3-dimensional truncation $\mathbf{3}\mathsf{Span}(\mathbf{C})$ consists of $\mathbf{i}$-cells for $\mathbf{i} \subset \mathbf{3} = \{0, 1, 2\}$ (or i-cubes and i-maps for $i < 3$, in the cubical framework).

Similarly one can find in [G1] the construction of the weak symmetric cubical category $\mathsf{Cosp}(\mathbf{C})$ *of cubical cospans* over a category $\mathbf{C}$ with (a fixed choice of) pushouts. An n-cube is now a functor $x\colon \wedge^n \to \mathbf{C}$, where $\wedge = \vee^{\mathrm{op}}$ is the formal-cospan category $0 \to u \leftarrow 1$; again, a transversal map of n-cubes is a natural transformation of such functors. $\mathsf{Cosp}(\mathbf{C})$ is *transversally dual* to $\mathsf{Span}(\mathbf{C}^{\mathrm{op}})$.

The 2-dimensional and 3-dimensional truncations are written as:

$$\mathbb{C}\mathrm{osp}(\mathbf{C}) = \mathbf{2}\mathsf{Cosp}(\mathbf{C}), \qquad\qquad \mathbf{3}\mathsf{Cosp}(\mathbf{C}). \tag{9}$$

1.3. The chiral case. Chiral multiple categories *of non-cubical type* are constructed in [GP7] and Part I, Section 4.

For instance, if the category $\mathbf{C}$ has pullbacks and pushouts, the weak double category $\mathbb{S}\mathrm{pan}(\mathbf{C})$, of arrows and spans of $\mathbf{C}$, can be 'amalgamated' with the weak double category $\mathbb{C}\mathrm{osp}(\mathbf{C})$, of arrows and cospans of $\mathbf{C}$, to form a 3-dimensional structure: the chiral triple category $\mathsf{SC}(\mathbf{C})$ whose 0-, 1- *and* 2-directed arrows are the arrows, spans and cospans of $\mathbf{C}$, *in this order* (as required by the 12-interchanger).

The highest cubes, of type $\{1,2\}$, are functors $x\colon \vee\times\wedge \to \mathbf{C}$, the highest (3-dimensional) cells are the natural transformations of the latter

$$\begin{array}{ccccc} (0,0) & \leftarrow & (u,0) & \rightarrow & (1,0) \\ \downarrow & & \downarrow & & \downarrow \\ (0,u) & \leftarrow & (u,u) & \rightarrow & (1,u) \\ \uparrow & & \uparrow & & \uparrow \\ (0,1) & \leftarrow & (u,1) & \rightarrow & (1,1) \end{array} \qquad \vee\times\wedge. \qquad\qquad \begin{array}{l} \bullet \xrightarrow{\ \ } {}^{1} \\ \big\downarrow{}_{2} \end{array} \tag{10}$$

Here 0-composition works by ordinary composition in $\mathbf{C}$, 1-composition by composing spans (with pullbacks) and 2-composition by composing cospans (with pushouts).

Higher dimensional examples, like $\mathsf{S}_p\mathsf{C}_q(\mathbf{C})$, $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$ and $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ (and the corresponding *left-chiral* cases) can be found in I.4.4; note that $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$ is indexed *by all integers*, with spans in each negative direction, ordinary arrows in direction 0 and cospans in positive directions.

1.4. A double adjunction. Let $\mathbf{C}$ be a category with distinguished pullbacks and pushouts. For the sake of simplicity we assume that the distinguished pullback (resp. pushout) of an identity along any map is an identity.

The weak double categories $\mathbb{S}\mathrm{pan}(\mathbf{C})$ and $\mathbb{C}\mathrm{osp}(\mathbf{C})$ of spans and cospans of $\mathbf{C}$ are linked by an obvious colax/lax adjunction

$$F\colon \mathbb{S}\mathrm{pan}(\mathbf{C}) \rightleftarrows \mathbb{C}\mathrm{osp}(\mathbf{C}) : G, \qquad \eta\colon 1 \dashrightarrow GF, \qquad \varepsilon\colon FG \dashrightarrow 1, \tag{11}$$

that we describe here *in an informal way.* (Writing $\eta\colon 1 \dashrightarrow GF$ and $\varepsilon\colon FG \dashrightarrow 1$ is an abuse of notation, since the comparisons of F and G have conflicting directions and cannot be composed. The precise definition of a colax/lax adjunction of weak double categories can be found in [GP2]; but the reader will find here its multiple extension, in Section 4, and can easily recover the truncated notion.)

At the basic level of $\mathrm{tv}_*(\mathbb{S}\mathrm{pan}(\mathbf{C})) = \mathrm{tv}_*(\mathbb{C}\mathrm{osp}(\mathbf{C})) = \mathbf{C}$ everything is an identity. At the level 1 (of 1-cubes and 1-maps) F operates by pushout and G by pullback; the special transversal 1-maps $\eta x\colon x \to GFx$ and $\varepsilon y\colon FGy \to y$ are obvious (for a span $x = (x', x'')$

and a cospan $y = (y', y'')$ with 1-faces A and B):

$$
\begin{tikzcd}
 & & A \arrow[dr] & \\
X \arrow[urr, "x'"] \arrow[drr, "x''"'] \arrow[r, dashed, "\eta x"] & \bullet \arrow[ur] \arrow[dr] & & \bullet \\
 & & B \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dr] \arrow[drr, "y'"] & & \\
\bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[r, dashed, "\varepsilon y"] & Y \\
 & B \arrow[ur] \arrow[urr, "y''"'] & &
\end{tikzcd}
\tag{12}
$$

The triangle identities are plainly satisfied:

$$
\varepsilon(Fx).F(\eta x) = \mathrm{id}(Fx), \qquad\qquad G(\varepsilon y).\eta(Gy) = \mathrm{id}(Gy).
$$

Finally it is easy to check that F is, in a natural way, a colax double functor (while G is lax). The comparison cell $\underline{F}(x, y) \colon F(x +_1 y) \to Fx +_1 Fy$ for concatenation is given by the natural map from the pushout of $x +_1 y = (x'z', y''z'')$ to the cospan $Fx +_1 Fy$ (with vertex Z' in the diagram below)

$$
\begin{tikzcd}
 & & A \arrow[dr] & & \\
 & X \arrow[ur, "x'"] \arrow[dr, "x''"] & & \bullet \arrow[dr] & \\
Z \arrow[ur, "z'"] \arrow[dr, "z''"'] & & B \arrow[ur] \arrow[dr] & & Z' \\
 & Y \arrow[ur, "y'"'] \arrow[dr, "y''"'] & & \bullet \arrow[ur] & \\
 & & C \arrow[ur] & &
\end{tikzcd}
\tag{13}
$$

Since we agreed to follow the unitarity constraint for the choice of pullbacks and pushouts in $\mathbf{C}$, the adjunction is *unitary*, in the sense that this property holds for the weak double categories $\mathbb{S}\mathrm{pan}(\mathbf{C})$, $\mathbb{C}\mathrm{osp}(\mathbf{C})$ and the colax/lax double functors F, G. It is also interesting to note that the restricted adjunction at the $\star$-level

$$
F_* \colon \mathbf{C} \rightleftarrows \mathbf{C} : G_* \qquad\qquad \eta_* \colon 1 \to G_*F_*, \qquad \varepsilon_* \colon F_*G_* \to 1, \tag{14}
$$

is composed of identity functors and identity transformations.

We also remark that the natural transformations $F\eta$, εF, ηG, $G\varepsilon$ at level 1 are invertible (which means that the ordinary adjunction at level 1 is idempotent: see [AT, LS]).

1.5. A MULTIPLE ADJUNCTION. Following [G3], the unitary colax double functor $F \colon \mathbb{S}\mathrm{pan}(\mathbf{C}) \to \mathbb{C}\mathrm{osp}(\mathbf{C})$ can be extended to a unitary colax multiple functor of cubical type

$$
F \colon \mathsf{Span}(\mathbf{C}) \to \mathsf{Cosp}(\mathbf{C}).
$$

For instance, let us take a 2-dimensional span $x \in \mathrm{Span}_{\mathbf{i}}(\mathbf{C})$ indexed by $\mathbf{i} = \{i, j\}$, as in the left diagram below

$$\begin{tikzcd} x(0,0) & x(u,0) \arrow[l] \arrow[r] & x(1,0) \\ x(0,u) \arrow[u] \arrow[d] & x(u,u) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & x(1,u) \arrow[u] \arrow[d] \\ x(0,1) & x(u,1) \arrow[l] \arrow[r] & x(1,1) \end{tikzcd} \qquad \begin{tikzcd} x(0,0) \arrow[r] \arrow[d] & F(\partial_j^- x)(u) \arrow[d] & x(1,0) \arrow[l] \arrow[d] \\ F(\partial_i^- x)(u) \arrow[r] & \mathrm{colim}(x) & F(\partial_i^+ x)(u) \arrow[l] \\ x(0,1) \arrow[r] \arrow[u] & F(\partial_j^+ x)(u) \arrow[u] & x(1,1) \arrow[l] \arrow[u] \end{tikzcd} \tag{15}$$

The 2-dimensional cospan $F(x) = F_{\mathbf{i}}(x)$ is constructed at the right hand, with the pushouts $F(\partial_i^\alpha x)$, $F(\partial_j^\alpha x)$ of the four faces and, in the central vertex, the colimit of the whole diagram $x\colon \vee^2 \to \mathbf{C}$. (The latter can be constructed in $\mathbf{C}$ as a pushout of pushouts; a general characterisation of the dual topic, limits 'generated' by pullbacks, can be found in [P1].)

One proceeds in a similar way, defining $F_{\mathbf{i}}$ for a positive multi-index $\mathbf{i} = \{i_1, ..., i_n\}$ of degree n, after all instances of degree $n-1$

$$\begin{aligned} &\partial_i^\alpha(F_{\mathbf{i}}(x)) = F_{\mathbf{i}|i}(\partial_i^\alpha x) && \text{for } \alpha = \pm \text{ and } i \in \mathbf{i}, \\ &F_{\mathbf{i}}(x)(\underline{u}) = \mathrm{colim}(x), && \text{where } \underline{u} = (u, ..., u) \in \vee^n. \end{aligned} \tag{16}$$

The definition of F on transversal $\mathbf{i}$-maps is obvious, as well as the comparison cells for the i-directed concatenation $\underline{F}_i(x, y)\colon F(x +_i y) \to Fx +_i Fy$.

The unitary lax double functor $G\colon \mathbb{C}\mathrm{osp}(\mathbf{C}) \to \mathbb{S}\mathrm{pan}(\mathbf{C})$ is similarly extended, using distinguished limits instead of colimits, and gives a unitary lax multiple functor $G\colon \mathsf{Cosp}(\mathbf{C}) \to \mathsf{Span}(\mathbf{C})$ of cubical type.

One extends the unit $\eta\colon 1 \dashrightarrow GF$ by a similar inductive procedure:

$$\begin{aligned} &\partial_i^\alpha(\eta_{\mathbf{i}}(x)) = \eta_{\mathbf{i}|i}(\partial_i^\alpha x) && \text{for } \alpha = \pm \text{ and } i \in \mathbf{i}, \\ &(\eta_{\mathbf{i}} x)(\underline{u})\colon x(\underline{u}) \to (G_{\mathbf{i}} F_{\mathbf{i}} x)(\underline{u}) = \lim(F_{\mathbf{i}} x), \end{aligned} \tag{17}$$

where the map $(\eta_{\mathbf{i}} x)(\underline{u})$ is given by the universal property of the limit $\lim(F_{\mathbf{i}} x)$ of the cubical cospan $F_{\mathbf{i}} x\colon \wedge^n \to \mathbf{C}$.

Analogously for the counit $\varepsilon\colon FG \dashrightarrow 1$. The triangular identities hold.

1.6. Chiral examples. The colax/lax adjunction of weak triple categories of cubical type

$$F\colon \mathbf{3}\mathsf{Span}(\mathbf{C}) \rightleftarrows \mathbf{3}\mathsf{Cosp}(\mathbf{C}) : G, \tag{18}$$

can be factorised through the chiral triple category $\mathsf{SC}(\mathbf{C})$ *of spans and cospans of* $\mathbf{C}$, obtaining two colax/lax adjunctions of chiral triple categories (no longer of cubical type)

$$F'\colon \mathbf{3}\mathsf{Span}(\mathbf{C}) \rightleftarrows \mathsf{SC}(\mathbf{C}) : G', \qquad F''\colon \mathsf{SC}(\mathbf{C}) \rightleftarrows \mathbf{3}\mathsf{Cosp}(\mathbf{C}) : G''. \tag{19}$$

Here the functor $F'\colon \mathbf{3}\mathsf{Span}(\mathbf{C}) \to \mathsf{SC}(\mathbf{C})$ acts on a 12-cube x

- by pushout on the three 2-directed spans of x,
- as the identity on the two 1-directed boundary spans $\partial_2^\alpha(x)$,
- by induced morphisms on the middle 1-directed span.

On the other hand the functor $G'\colon \mathsf{SC}(\mathbf{C}) \to \mathbf{3}\mathsf{Span}(\mathbf{C})$ acts by pullback on the three (2-directed) cospans of x, as the identity on the (1-directed) boundary spans $\partial_2^\alpha(x)$ and by induced morphisms on the middle span. Similarly for F'' and G''.

One can also factorise the adjunction (18) through the *left* chiral triple category $\mathsf{CS}(\mathbf{C})$ *of cospans and spans*, obtaining two colax/lax adjunctions of left chiral triple categories.

Similarly, the multiple adjunction constructed in 1.5 can be factorised through any right chiral multiple category $\mathsf{S}_p\mathsf{C}_\infty(\mathbf{C})$, *or* through any left chiral multiple category $\mathsf{C}_p\mathsf{S}_\infty(\mathbf{C})$.

However, in infinite dimension, one may prefer to consider a more symmetric situation, starting from a colax/lax adjunction of weak multiple categories *indexed by the ordered set* $\mathbb{Z}$ of integers (pointed at 0)

$$F\colon \mathbb{Z}\mathsf{Span}(\mathbf{C}) \rightleftarrows \mathbb{Z}\mathsf{Cosp} : G. \tag{20}$$

This can be factorised through the chiral multiple category $\mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C})$, obtaining two colax/lax adjunctions of 'unbounded' chiral multiple categories

$$F'\colon \mathbb{Z}\mathsf{Span}(\mathbf{C}) \rightleftarrows \mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C}) : G', \qquad F''\colon \mathsf{S}_{-\infty}\mathsf{C}_\infty(\mathbf{C}) \rightleftarrows \mathbb{Z}\mathsf{Cosp}(\mathbf{C}) : G''. \tag{21}$$

1.7. Other examples. Now we start from an ordinary adjunction $F \dashv G$

$$F\colon \mathbf{X} \rightleftarrows \mathbf{A} : G, \qquad \eta\colon 1 \to GF, \qquad \varepsilon\colon FG \to 1, \tag{22}$$

between categories with (a choice of) pullbacks. This can be extended in a natural way to a unitary *colax/pseudo* adjunction between the weak multiple categories of higher spans (of cubical type)

$$\mathsf{Span}(F)\colon \mathsf{Span}(\mathbf{X}) \rightleftarrows \mathsf{Span}(\mathbf{A}) : \mathsf{Span}(G). \tag{23}$$

In fact there is an obvious 2-functor

$$\mathsf{Span}\colon \mathbf{Cat}_{\mathrm{pb}} \to \mathrm{Cx}\mathbf{Cmc}, \tag{24}$$

defined on the full sub-2-category of $\mathbf{Cat}$ containing all categories with (a choice of) pullbacks, with values in the 2-category of chiral multiple categories, colax functors and their transversal transformations (see 2.1).

It sends a category $\mathbf{C}$ with pullbacks to the chiral multiple category $\mathsf{Span}(\mathbf{C})$ (actually a weak multiple category of symmetric cubical type).

For a functor $F\colon \mathbf{X} \to \mathbf{A}$ (between categories with pullbacks), $\mathsf{Span}(F)$, also written as F for brevity, simply acts by computing F over the diagrams of $\mathbf{X}$ that form $\mathbf{i}$-cubes and $\mathbf{i}$-maps; formally, over an $\mathbf{i}$-map $f\colon x \to y\colon \vee^n \to \mathbf{X}$, $F(f)\colon F(x) \to F(y)$ is the composite $F.f\colon F.x \to F.y\colon \vee^n \to \mathbf{A}$. This extension is, in a natural way, a unitary *colax* functor, since identities of $\mathbf{X}$ go to identities of $\mathbf{A}$ and a composition $x +_i y$ of two spans $x, y\colon \vee \to \mathbf{X}$ (in any direction $i > 0$) gives rise to a diagram in $\mathbf{X}$ and a diagram in $\mathbf{A}$

$$\begin{tikzcd} & & X_1 \\ & X \arrow[ur, "x'"] \arrow[dr, "x''"] & \\ P \arrow[ur, "z'"] \arrow[dr, "z''"'] & & X_2 \\ & Y \arrow[ur, "y'"'] \arrow[dr, "y''"'] & \\ & & X_3 \end{tikzcd} \qquad \begin{tikzcd} & & & FX_1 \\ & & FX \arrow[ur, "Fx'"] \arrow[dr, "Fx''"] & \\ FP \arrow[r, dashed, "a"] \arrow[urr, "Fz'"] \arrow[drr, "Fz''"'] & Q \arrow[ur] \arrow[dr] & & FX_2 \\ & & FY \arrow[ur, "Fy'"'] \arrow[dr, "Fy''"'] & \\ & & & FX_3 \end{tikzcd} \tag{25}$$

where the comparison $\underline{F}_i(x,y)\colon F(x +_i y) \to F(x) +_i F(y)$ is an i-special transversal map given by the $\mathbf{A}$-morphism $a\colon FP \to Q$ determined by the universal property of the pullback Q. Similarly we define $\underline{F}_i(x,y)$ for every i-composition of $\mathbf{i}$-cubes. Note that $\mathsf{Span}(F)$ is pseudo if (and only if) the functor $F\colon \mathbf{X} \to \mathbf{A}$ preserves pullbacks.

Finally, a natural transformation $\varphi\colon F \to F'\colon \mathbf{X} \to \mathbf{A}$ yields a transversal transformation

$$\mathsf{Span}(\varphi)\colon \mathsf{Span}(F) \to \mathsf{Span}(F')\colon \mathsf{Span}(\mathbf{X}) \to \mathsf{Span}(\mathbf{A}), \tag{26}$$

that again will often be written as φ. On an $\mathbf{i}$-cube $x\colon \vee^n \to \mathbf{X}$, $\varphi x\colon F(x) \to_0 F'(x)$ is the composite of the functor $x\colon \vee^n \to \mathbf{X}$ with the natural transformation $\varphi\colon F \to F'\colon \mathbf{X} \to \mathbf{A}$. Concretely, the transversal $\mathbf{i}$-map $\varphi x\colon F(x) \to_0 F'(x)$ has components $\varphi(x(t))$, for every vertex t of $\vee^n$.

Now, letting the 2-functor $\mathsf{Span}\colon \mathbf{Cat}_{\mathrm{pb}} \to \mathrm{Cx}\mathbf{Cmc}$ act on the adjunction (22), we get an adjunction of weak multiple categories with the properties stated above: $\mathsf{Span}(F)$ is colax, $\mathsf{Span}(G)$ is pseudo and both are unitary.

On the other hand, if $\mathbf{X}$ and $\mathbf{A}$ have pushouts, the adjunction (22) yields a *pseudo/lax* adjunction of weak multiple categories

$$F\colon \mathsf{Cosp}(\mathbf{X}) \rightleftarrows \mathsf{Cosp}(\mathbf{A}) : G. \tag{27}$$

Finally, if $\mathbf{X}$ and $\mathbf{A}$ have pullbacks and pushouts, we can extend (22) to a colax/lax adjunction of chiral triple categories

$$F\colon \mathsf{SC}(\mathbf{X}) \rightleftarrows \mathsf{SC}(\mathbf{A}) : G, \tag{28}$$

or consider higher-dimensional extensions of 'type' $\mathsf{S}_p\mathsf{C}_q$, $\mathsf{S}_p\mathsf{C}_\infty$, $\mathsf{S}_{-\infty}\mathsf{C}_\infty$, etc. (see 1.3). Note that, according to the analysis of [GP6], Section 5, F is a *colax-pseudo* morphism of chiral triple categories (i.e. colax for the 1-directed composition, realised by pullbacks, and pseudo for the 2-composition, realised by pushouts) while G is pseudo-lax.

2. The double category of lax and colax multiple functors

In the 2-dimensional case, weak double categories with lax and colax double functors and suitable double cells form a *strict* double category $\mathbb{D}\mathrm{bl}$, a crucial structure introduced in [GP2] to define colax/lax double adjunctions - also recalled in Part I.

This construction was extended in [G3], Section 2, to form the strict *double* category $\mathbb{W}\mathrm{sc}$ of weak symmetric cubical categories, lax and colax symmetric cubical functors (and suitable double cells), in order to define colax/lax adjunctions between weak symmetric cubical categories.

We now give a further extension, building the strict *double* category $\mathbb{C}\mathrm{mc}$ of chiral multiple categories, lax and colax multiple functors, that will be used below to define colax/lax adjunctions between chiral multiple categories.

Comma chiral multiple categories are also considered, extending again the cases of double and symmetric cubical categories, dealt with in [GP2, G3].

For $\mathbb{C}\mathrm{mc}$ we follow the notation for double categories used in [GP1] - [GP4]: the horizontal and vertical compositions of cells are written as $(\pi \mid \rho)$ and $(\frac{\pi}{\sigma})$, or more simply as $\pi|\rho$ and $\pi \otimes \sigma$. Horizontal identities, of an object or a vertical arrow, are written as 1_A and 1_u; vertical identities, of an object or a horizontal arrow, as $1^\bullet_A$ and $1^\bullet_f$.

2.1. Lax cm-functors. A chiral multiple category is also called a *cm-category*, for short.

As defined in I.3.9, a *lax multiple functor* $F\colon \mathsf{A} \to \mathsf{B}$ between chiral multiple categories, or *lax cm-functor*, has components $F_\mathbf{i}\colon A_\mathbf{i} \to B_\mathbf{i}$ for all multi-indices $\mathbf{i}$ (often written as F) that agree with all faces, 0-degeneracies and 0-composition. Moreover, for every positive multi-index $\mathbf{i}$ and $i \in \mathbf{i}$, F is equipped with i-special comparison $\mathbf{i}$-maps $\underline{F}_i$ that agree with faces

$$\begin{aligned}
&\underline{F}_i(x)\colon e_iF(x) \to_0 F(e_ix) && (\text{for } x \text{ in } A_{\mathbf{i}|i})\\
&\underline{F}_i(x,y)\colon F(x) +_i F(y) \to_0 F(z) && (\text{for } z = x +_i y \text{ in } A_\mathbf{i}), \qquad (29)\\
&\partial^\alpha_j\underline{F}_i(x) = \underline{F}_i(\partial^\alpha_j x), \qquad \partial^\alpha_j\underline{F}_i(x,y) = \underline{F}_i(\partial^\alpha_j x, \partial^\alpha_j y) && (\text{for } j \neq i).
\end{aligned}$$

and satisfy some axioms. We write down the naturality conditions (lmf.1-2), frequently used below, while the coherence conditions (lmf.3-5) can be found in I.3.9.

(lmf.1) (*naturality of unit comparisons*) For every $\mathbf{i}|j$-map $f\colon x \to_0 y$ in A we have:

$$F(e_jf).\underline{F}_j(x) = \underline{F}_j(y).e_j(Ff)\colon e_jF(x) \to_0 F(e_jy). \qquad (30)$$

(lmf.2) (*naturality of composition comparisons*) For two j-consecutive $\mathbf{i}$-maps $f\colon x \to_0 x'$ and $g\colon y \to_0 y'$ in A we have:

$$F(f +_j g).\underline{F}_j(x,y) = \underline{F}_j(x',y').(F(f) +_j F(g))\colon F(x) +_j F(y) \to_0 F(x' +_j y'). \qquad (31)$$

A *transversal transformation* $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ between lax multiple functors of chiral multiple categories consists of a face-consistent family of $\mathbf{i}$-maps in B (its components), one for every positive multi-index $\mathbf{i}$ and every $\mathbf{i}$-cube x in A

$$hx\colon F(x) \to_0 G(x), \qquad h(\partial^\alpha_j x) = \partial^\alpha_j(hx), \qquad (32)$$

under the axioms (trt.1) and (trt.2L), see I.3.9.

We have thus the 2-category Lx**Cmc** of cm-categories, lax cm-functors and their transversal transformations. Similarly one defines the 2-category Cx**Cmc**, for the colax case where the comparisons of *colax cm-functors* have the opposite direction. A *pseudo cm-functor* is a lax cm-functor whose comparisons are invertible; it is made colax by the inverse comparisons.

2.2. The double category $\mathbb{C}$mc. Lax and colax cm-functors do not compose well, since we cannot compose their comparisons. But they give the horizontal and vertical arrows of a *strict* double category $\mathbb{C}$mc, crucial for our study, where 'internal' orthogonal adjunctions (recalled below, in Section 3) will provide our general notion of multiple adjunction (Section 4) while companion pairs amount to pseudo cm-functors (Section 5).

The objects of $\mathbb{C}$mc are the cm-categories $\mathsf{A}, \mathsf{B}, \mathsf{C}, ...$; its horizontal arrows are the *lax* cm-functors $R, S...$; its vertical arrows are the *colax* cm-functors $F, G...$

A double cell $\pi\colon (F\, {}^{R}_{S}\, G)$

$$\begin{tikzcd} \mathsf{A} \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\pi"] & \mathsf{B} \arrow[d, "G"] \\ \mathsf{C} \arrow[r, "S"'] & \mathsf{D} \end{tikzcd} \tag{33}$$

is - roughly speaking - a 'transformation' $\pi\colon GR \dashrightarrow SF$. But the composites GR and SF are neither lax nor colax: the coherence conditions of π require the individual knowledge of the four 'functors', including the comparison cells of each of them.

Precisely, the double cell π consists of the following data:

(a) two lax cm-functors R, S, with comparisons as follows:

$$R\colon \mathsf{A} \to \mathsf{B}, \quad \underline{R}_i(x)\colon e_i(Rx) \to_0 R(e_ix), \quad \underline{R}_i(x,y)\colon Rx +_i Ry \to_0 R(x +_i y),$$
$$S\colon \mathsf{C} \to \mathsf{D}, \quad \underline{S}_i(x)\colon e_i(Sx) \to_0 S(e_ix), \quad \underline{S}_i(x,y)\colon Sx +_i Sy \to_0 S(x +_i y),$$

(b) two colax cm-functors F, G with comparisons as follows:

$$F\colon \mathsf{A} \to \mathsf{C}, \quad \underline{F}_i(x)\colon F(e_ix) \to_0 e_i(Fx), \quad \underline{F}_i(x,y)\colon F(x +_i y) \to_0 Fx +_i Fy,$$
$$G\colon \mathsf{B} \to \mathsf{D}, \quad \underline{G}_i(x)\colon G(e_ix) \to_0 e_i(Gx), \quad \underline{G}_i(x,y)\colon G(x +_i y) \to_0 Gx +_i Gy,$$

(c) a family of $\mathbf{i}$-maps $\pi x\colon GR(x) \to_0 SF(x)$ of D (for every $\mathbf{i}$-cube x in A), consistent with faces

$$\pi(\partial_i^\alpha x) = \partial_i^\alpha(\pi x). \tag{34}$$

These data have to satisfy the naturality condition (dc.1) and the coherence conditions (dc.2), (dc.3) (with respect to i-degeneracies and i-composition, respectively)

(dc.1) $SFf.\pi x = \pi y.GRf\colon GR(x) \to_0 SF(y)$ (for $f\colon x \to_0 y$ in $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$),

(dc.2) $S\underline{F}_i(x).\pi e_i(x).G\underline{R}_i(x) = \underline{S}_iF(x).e_i(\pi x).\underline{G}_iR(x)$ (for x in $A_{\mathbf{i}|i}$),

$$\begin{tikzcd}
Ge_i(Rx) \arrow[r, "G\underline{R}_i(x)"] \arrow[d, "\underline{G}_iR(x)"'] & GR(e_ix) \arrow[r, "\pi e_i(x)"] & SF(e_ix) \arrow[d, "S\underline{F}_i(x)"] \\
e_iGR(x) \arrow[r, "e_i\pi(x)"'] & e_iSF(x) \arrow[r, "\underline{S}_iF(x)"'] & Se_iF(x)
\end{tikzcd}$$

(dc.3) $S\underline{F}_i(x,y).\pi z.G\underline{R}_i(x,y) = \underline{S}_i(Fx,Fy).(\pi x{+_i}\pi y).\underline{G}_i(Rx,Ry)$ (for $z = x{+_i}y$ in $A_{\mathbf{i}}$),

$$\begin{tikzcd}
G(Rx +_i Ry) \arrow[r, "G\underline{R}_i(x,y)"] \arrow[d, "\underline{G}_i(Rx,Ry)"'] & GR(z) \arrow[r, "\pi z"] & SF(z) \arrow[d, "S\underline{F}_i(x,y)"] \\
GRx +_i GRy \arrow[r, "\pi x+_i\pi y"'] & SFx +_i SFy \arrow[r, "\underline{S}_i(Fx,Fy)"'] & S(Fx +_i Fy)
\end{tikzcd}$$

The horizontal composition $(\pi\,|\,\rho)$ and the vertical composition $\pi\otimes\sigma = (\frac{\pi}{\sigma})$ of double cells are both defined *via the composition of transversal maps* (in a cm-category)

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "R"] \arrow[d, "F"'] \arrow[dr, phantom, "\pi"] & \bullet \arrow[r, "R'"] \arrow[d, "G"] \arrow[dr, phantom, "\rho"] & \bullet \arrow[d, "H"] \\
\bullet \arrow[r, "S"] \arrow[d, "F'"'] \arrow[dr, phantom, "\sigma"] & \bullet \arrow[r, "S'"] \arrow[d, "G'"] \arrow[dr, phantom, "\tau"] & \bullet \arrow[d, "H'"] \\
\bullet \arrow[r, "T"'] & \bullet \arrow[r, "T'"'] & \bullet
\end{tikzcd} \tag{35}$$

$$\begin{aligned}
&(\pi\,|\,\rho)(x) = S'\pi x.\rho Rx\colon\, HR'R(x) \to_0 S'GR(x) \to_0 S'SF(x),\\
&(\tfrac{\pi}{\sigma})(x) = \sigma Fx.G'\pi x\colon\, G'GR(x) \to_0 G'SF(x) \to_0 TF'F(x) \qquad \text{(for } x \text{ in } \mathsf{A}).
\end{aligned} \tag{36}$$

We verify below, in Theorem 2.3, that these compositions are well-defined and satisfy the axioms of a double category.

Within $\mathbb{C}$mc, we have the strict 2-category Lx**Cmc** of *cm-categories, lax cm-functors and transversal transformations*: namely, Lx**Cmc** is the restriction of $\mathbb{C}$mc to trivial vertical arrows. Symmetrically, the strict 2-category Cx**Cmc**, whose arrows are the colax cm-functors, also lies in $\mathbb{C}$mc.

As in I.1.2 (for weak double categories), we can also note that a double cell $\pi\colon ({}_F\,{}^R_1\,{}_1)$ gives a notion of *transversal transformation* $\pi\colon R \dashrightarrow F\colon \mathsf{A}\to\mathsf{B}$ *from a lax to a colax functor*, while a double cell $\pi\colon ({}_1\,{}^1_S\,{}_G)$ gives a notion of *transversal transformation* $\pi\colon G \dashrightarrow S\colon \mathsf{A}\to\mathsf{B}$ *from a colax to a lax functor*. Moreover, for a fixed pair A,B of chiral multiple categories, all the transversal transformations between lax and colax functors (of the four possible kinds) compose, forming a category $\{\mathsf{A},\mathsf{B}\}$ whose objects are the lax *and* the colax functors $\mathsf{A}\to\mathsf{B}$.

2.3. Theorem. $\mathbb{C}$mc, *as defined above, is indeed a strict double category.*

Proof. The argument is much the same as for $\mathbb{D}$bl in [GP2] and for $\mathbb{W}$sc in [G3].

First, to show that the double cells defined in (36) are indeed coherent, we verify the condition (dc.3) for $(\pi \,|\, \rho)$, with respect to a concatenation $z = x +_i y$ in A. Our property amounts to the commutativity of the outer diagram below, formed of transversal maps (where we omit the index i in $+_i$ and all comparisons $\underline{R}_i$ etc.)

$$\begin{tikzcd}
HR'Rz \arrow[r, "\rho Rz"] & S'GRz \arrow[r, "S'\pi z"] & S'SFz \arrow[d, "S'S\underline{F}"] \\
HR'(Rx+Ry) \arrow[u, "HR'\underline{R}"] \arrow[r, "\rho(Rx+Ry)"] & S'G(Rx+Ry) \arrow[u, "S'G\underline{R}"'] \arrow[d, "S'\underline{G}R"] & S'S(Fx+Fy) \\
H(R'Rx+R'Ry) \arrow[u, "H\underline{R}'R"] \arrow[d, "\underline{H}R'R"] & S'(GRx+GRy) \arrow[r, "S'(\pi x+\pi y)"] & S'(SFx+SFy) \arrow[u, "S'\underline{S}F"'] \\
HR'Rx+HR'Ry \arrow[r, "\rho Rx+\rho Ry"'] & S'GRx+S'GRy \arrow[u, "\underline{S}'GR"'] \arrow[r, "S'\pi x+S'\pi y"'] & S'SFx+S'SFy \arrow[u, "\underline{S}'SF"']
\end{tikzcd}$$

Indeed, the two hexagons commute applying (dc.3) to π and ρ, respectively. The upper square commutes by naturality of ρ on $\underline{R}_i(x, y)$; the lower one by axiom (lmf.2) (see 2.1) on the lax functor S', with respect to the **i**-maps $\pi x\colon GR(x) \to_0 SF(x)$ and $\pi y\colon GR(y) \to_0 SF(y)$

$$S'(\pi x +_i \pi y).\underline{S}'_i(GR(x), GR(y)) = \underline{S}'_i(SF(x), SF(y)).(S'(\pi x) +_i S'(\pi y)).$$

Now, both composition laws of double cells have been defined, in (36), *via the composition of transversal maps* (in a cm-category), and therefore are strictly unitary and associative.

Finally, to verify the middle-four interchange law on the four double cells of diagram (35), we compute the compositions $(\pi \,|\, \rho) \otimes (\sigma \,|\, \tau)$ and $(\pi \otimes \sigma) \,|\, (\rho \otimes \tau)$ on an **i**-cube x, and we obtain the two transversal maps $H'HR'Rx \to_0 T'TF'Fx$ of the upper or lower path in the following diagram

$$\begin{tikzcd}
H'HR'Rx \arrow[r, "H'\rho Rx"] & H'S'GRx \arrow[r, "H'S'\pi x"] \arrow[d, "\tau GRx"'] & H'S'SFx \arrow[d, "\tau SFx"] & \\
& T'G'GRx \arrow[r, "T'G'\pi x"'] & T'G'SFx \arrow[r, "T'\sigma Fx"'] & T'TF'Fx
\end{tikzcd}$$

But these two composites coincide because the square commutes: a consequence of the naturality of τ on the transversal map $\pi x\colon GR(x) \to_0 SF(x)$, by axiom (dc.1) for the double cell τ. ∎

2.4. Comma cm-categories. Comma double categories [GP2] also have a natural extension to the multiple case.

Given a *colax* cm-functor $F\colon \mathsf{A} \to \mathsf{C}$ and a *lax* cm-functor $R\colon \mathsf{X} \to \mathsf{C}$ with the same codomain, we can construct the *comma cm-category* $F{\downarrow}R$, where the projections P and Q are strict cm-functors and π is a double cell of $\mathbb{C}$mc

$$\begin{tikzcd} F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\pi"] & \mathsf{A} \arrow[d, "F"] \\ \mathsf{X} \arrow[r, "R"'] & \mathsf{C} \end{tikzcd} \tag{37}$$

An **i**-cube of $F{\downarrow}R$ is a triple $(a, x; c\colon Fa \to_0 Rx)$ where a is an **i**-cube of A, x is an **i**-cube of X and c is an **i**-map of C. An **i**-map $(h, f)\colon (a, x; c) \to_0 (a', x'; c')$ comes from a pair of **i**-maps $h\colon a \to_0 a'$ (in A) and $f\colon x \to_0 x'$ (in X) that give in C a commutative square of transversal maps

$$\begin{tikzcd} Fa \arrow[r, "c"] \arrow[d, "Fh"'] & Rx \arrow[d, "Rf"] \\ Fa' \arrow[r, "c'"'] & Rx' \end{tikzcd} \qquad\qquad Rf.c = c'.Fh. \tag{38}$$

Faces and transversal composition are obvious. The degeneracies are defined using the fact that F is *colax* and R is *lax*:

$$e_i(a, x; c\colon Fa \to_0 Rx) = (e_ia, e_ix; \underline{R}_i(x).e_ic.\underline{F}_i(a)). \tag{39}$$

Similarly the i-concatenation is defined as follows

$$\begin{aligned} &(a, x; c) \ +_i \ (b, y; d) \ = \ (a +_i b,\ x +_i y;\ u\colon F(a +_i b) \to R(x +_i y)),\\ &u = \underline{R}_i(x, y).(c +_i d).\underline{F}_i(a, b)\colon\\ &\qquad F(a +_i b) \ \to_0 \ Fa +_i Fb \ \to_0 \ Rx +_i Ry \ \to_0 \ R(x +_i y). \end{aligned} \tag{40}$$

The associativity comparison for the i-composition of three i-consecutive **i**-cubes of $F{\downarrow}R$

$$(a, x; c), \qquad\qquad (a', x'; c'), \qquad\qquad (a'', x''; c''),$$

is given by the pair $(\kappa_i(\mathbf{a}), \kappa_i(\mathbf{x}))$ of associativity **i**-isomorphisms for our two triples of **i**-cubes, namely $\mathbf{a} = (a, a', a'')$ in A and $\mathbf{x} = (x, x', x'')$ in X (we write $+_i$ as $+$)

$$\begin{aligned} &(\kappa_i(\mathbf{a}), \kappa_i(\mathbf{x}))\colon\\ &\quad (a, x; c) + ((a', x'; c') + (a'', x''; c'')) \to_0 ((a, x; c) + (a', x'; c')) + (a'', x''; c''). \end{aligned} \tag{41}$$

The coherence of this **i**-map, as in diagram (38) above, is proved in the lemma below.

Similarly one constructs the unitors λ_i, ρ_i and the interchangers χ_{ij} of $F{\downarrow}R$, using those of A and X.

Finally, the strict cm-functors P and Q are projections, while the component of the 'transformation' π on the **i**-cube $(a, x; c)$ of $F{\downarrow}R$ is the transversal map:

$$\pi(a, x; c) = c\colon Fa \to Rx. \tag{42}$$

2.5. Lemma. *The pair* $(\kappa(\mathbf{a}), \kappa(\mathbf{x}))$ *is indeed an* $\mathbf{i}$*-map of* $F{\downarrow}R$*, with domain and codomain as specified in* (41).

Proof. First, these two $\mathbf{i}$-cubes of $F{\downarrow}R$ can be written as:

$$
\begin{aligned}
(a, x; c) + ((a', x'; c') + (a'', x''; c'')) &= (a_1, x_1; c_1), \\
((a, x; c) + (a', x'; c')) + (a'', x''; c'') &= (a_2, x_2; c_2),
\end{aligned}
\tag{43}
$$

where (always leaving the index i understood for $+, \underline{F}, \underline{R}$)

$$
\begin{gathered}
a_1 = a + (a' + a''), \qquad\qquad x_1 = x + (x' + x''), \\
a_2 = (a + a') + a'', \qquad\qquad x_2 = (x + x') + x'', \\
c_1 = \underline{R}(x, x' + x'').(1 + \underline{R}(x', x'')).(c + (c' + c'')).(1 + \underline{F}(a', a'')).\underline{F}(a, a' + a''), \\
c_2 = \underline{R}(x + x', x'').(\underline{R}(x, x') + 1).((c + c') + c'').(\underline{F}(a, a') + 1).\underline{F}(a + a', a''),
\end{gathered}
\tag{44}
$$

$$
\begin{tikzcd}
Fa_1 \arrow[r, "c_1"] \arrow[d] & Rx_1 & Fa_2 \arrow[r, "c_2"] \arrow[d] & Rx_2 \\
Fa + F(a' + a'') \arrow[d] & Rx + R(x' + x'') \arrow[u] & F(a + a') + Fa'' \arrow[d] & R(x + x') + Rx'' \arrow[u] \\
Fa + (Fa' + Fa'') \arrow[r] & Rx + (Rx' + Rx'') \arrow[u] & (Fa + Fa') + Fa'' \arrow[r] & (Rx + Rx') + Rx'' \arrow[u]
\end{tikzcd}
$$

Now our claim, i.e. the condition for $(\kappa(\mathbf{a}), \kappa(\mathbf{x}))$ expressed in diagram (38), amounts to

$$
R\kappa(\mathbf{x}).c_1 = c_2.F\kappa(\mathbf{a})\colon Fa_1 \to Rx_2. \tag{45}
$$

First, the coherence of the lax functor R with the associator κ gives (applying axiom (lmf.4) of I.3.9):

$$
\begin{aligned}
&R\kappa(\mathbf{x}).c_1 \\
&\quad = R\kappa(\mathbf{x}).\underline{R}(x, x' + x'').(1 + \underline{R}(x', x'')).(c + (c' + c'')).(1 + \underline{F}(a', a'')).\underline{F}(a, a' + a'') \\
&\quad = \underline{R}(x + x', x'').(\underline{R}(x, x') + 1).\kappa R(\mathbf{x}).(c + (c' + c'')).(1 + \underline{F}(a', a'')).\underline{F}(a, a' + a'').
\end{aligned}
$$

Second, the coherence of the colax functor F with κ gives (applying the corresponding axiom, with reversed comparisons $\underline{F}$):

$$
\begin{aligned}
&c_2.F\kappa(\mathbf{a}) \\
&\quad = \underline{R}(x + x', x'').(\underline{R}(x, x') + 1).((c + c') + c'').(\underline{F}(a, a') + 1).\underline{F}(a + a', a'').F\kappa(\mathbf{a}) \\
&\quad = \underline{R}(x + x', x'').(\underline{R}(x, x') + 1).((c + c') + c'').\kappa(F\mathbf{a}).(1 + \underline{F}(a', a'')).\underline{F}(a, a' + a'').
\end{aligned}
$$

Finally, condition (45) follows from the naturality of κ on the triple of transversal maps $(c, c', c'')\colon F\mathbf{a} \to R\mathbf{x}$, which gives the commutative diagram

$$\begin{tikzcd}
Fa + (Fa' + Fa'') \arrow[r, "\kappa(F\mathbf{a})"] \arrow[d, "c+(c'+c'')"'] & (Fa + Fa') + Fa'' \arrow[d, "(c+c')+c''"] \\
Rx + (Rx' + Rx'') \arrow[r, "\kappa(R\mathbf{x})"'] & (Rx + Rx') + Rx''
\end{tikzcd}$$

■

2.6. Theorem. [Universal properties of commas] *(a) For a pair of* lax *cm-functors S, T and a cell φ as below (in $\mathbb{C}$mc) there is a unique lax cm-functor $L\colon \mathsf{Z} \to F{\downarrow}R$ such that $S = PL$, $T = QL$ and $\varphi = (\psi \,|\, \pi)$ where the cell ψ is defined by the identity $1\colon QL \to T$ (a horizontal transformation of lax cm-functors)*

$$\begin{tikzcd}
\mathsf{Z} \arrow[rr, "S"] \arrow[d, "1"'] \arrow[drr, phantom, "\varphi"] & & \mathsf{A} \arrow[d, "F"] \\
\mathsf{Z} \arrow[r, "T"'] & \mathsf{X} \arrow[r, "R"'] & \mathsf{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathsf{Z} \arrow[r, "L"] \arrow[d, "1"'] \arrow[dr, phantom, "\psi"] & F{\downarrow}R \arrow[r, "P"] \arrow[d, "Q"] \arrow[dr, phantom, "\pi"] & \mathsf{A} \arrow[d, "F"] \\
\mathsf{Z} \arrow[r, "T"'] & \mathsf{X} \arrow[r, "R"'] & \mathsf{C}
\end{tikzcd} \qquad (46)$$

Moreover L is pseudo or strict if and only if both S and T are.

(b) A similar property holds for a pair of colax *cm-functors G, H and a double cell $\varphi'\colon (G\,{}^{1}_{R}\,FH)$.*

Proof. (a) L is defined as follows on an $\mathbf{i}$-cube z and an $\mathbf{i}$-map $f\colon z \to z'$ of Z

$$L(z) = (Sz, Tz;\ \varphi z\colon FSz \to RTz), \qquad\qquad L(f) = (Sf, Tf). \qquad (47)$$

The comparison transversal maps $\underline{L}_i$ are constructed with the laxity transversal maps $\underline{S}$ and $\underline{T}$ (and are invertible or degenerate if and only if the latter are)

$$\begin{aligned}
&\underline{L}_i x = (\underline{S}_i x, \underline{T}_i x)\colon e_i(Lx) \to Le_i(x) && \text{(for } x \text{ in } Z_{\mathbf{i}|i}\text{)},\\
&\underline{L}_i(x, y) = (\underline{S}_i(x, y), \underline{T}_i(x, y))\colon Lx +_i Ly \to L(z) && \text{(for } z = x +_i y \text{ in } Z_{\mathbf{i}}\text{)}.
\end{aligned} \qquad (48)$$

Here $Lx +_i Ly$ is the $\mathbf{i}$-cube defined as below (applying (40))

$$\begin{aligned}
Lx +_i Ly &= (Sx, Tx;\ \varphi x\colon FSx \to RTx)\ +_i\ (Sy, Ty;\ \varphi y\colon FSy \to RTy)\\
&\quad = (Sx +_i Sy, Tx +_i Ty;\ u),\\
u &= \underline{R}_i(Tx, Ty).(\varphi x +_i \varphi y).\underline{F}_i Sx, Sy)\colon\\
&\quad F(Sx +_i Sy) \to FSx +_i FSy \to RTx +_i RTy \to R(Tx +_i Ty).
\end{aligned}$$

The coherence condition (38) on the transversal map $\underline{L}_i(x,y) = (\underline{S}_i(x,y), \underline{T}_i(x,y))$ of $F{\downarrow}R$ (with $z = x +_i y$)

$$\begin{tikzcd} F(Sx +_i Sy) \arrow[r, "u"] \arrow[d, "F\underline{S}_i(x,y)"'] & R(Tx +_i Ty) \arrow[d, "R\underline{T}_i(x,y)"] \\ FS(z) \arrow[r, "\varphi z"'] & RT(z) \end{tikzcd} \qquad R\underline{T}_i(x,y).u = \varphi z.F\underline{S}_i(x,y), \tag{49}$$

follows from the coherence condition (dc.3) of φ as a double cell in $\mathbb{C}$mc (where $\underline{RT}_i(x,y) = R\underline{T}_i(x,y).\underline{R}_i(Tx,Ty)$)

$$\underline{RT}_i(x,y).(\varphi x +_i \varphi y).\underline{F}_i(Sx,Sy) = \varphi z.F\underline{S}_i(x,y), \tag{50}$$

$$\begin{tikzcd} F(Sx +_i Sy) \arrow[r, "F\underline{S}_i(x,y)"] \arrow[d, "\underline{F}_i(Sx,Sy)"'] & FS(z) \arrow[r, "\varphi z"] & RT(z) \arrow[d, "1"] \\ FSx +_i FSy \arrow[r, "\varphi x +_i \varphi y"'] & RTx +_i RTy \arrow[r, "\underline{RT}_i(x,y)"'] & RT(z) \end{tikzcd}$$

The uniqueness of L is obvious. ■

3. Companions and adjoints in double categories

This brief section, taken from [GP2], Section 1, studies the connections between horizontal and vertical morphisms in a double category: horizontal morphisms can have vertical *companions* and vertical *adjoints* (the latter were called 'conjoints' in [DPR]). Such phenomena are interesting in themselves and typical of double categories.

$\mathbb{D}$ is always a weak double category, that we assume to be *unitary* for the sake of simplicity. We shall apply these notions to $\mathbb{C}$mc, which is strict.

3.1. Orthogonal companions. In the weak double category $\mathbb{D}$, the horizontal morphism $f\colon A \to B$ and the vertical morphism $u\colon A \to\!\!\!\bullet\, B$ are made (orthogonal) *companions* by assigning a pair (η, ε) of cells as below, called the *unit* and *counit*, that satisfy the identities $\eta|\varepsilon = 1^\bullet_f$ and $\eta \otimes \varepsilon = 1_u$

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "1"', "\bullet" marking] \arrow[dr, phantom, "\eta"] & A \arrow[d, "u", "\bullet" marking] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, "1", "\bullet" marking] \\ B \arrow[r, equal] & B \end{tikzcd} \tag{51}$$

Given f, this is equivalent (*by unitarity*) to saying that the pair (u, ε) satisfies the following universal property:

(a) for every cell $\varepsilon' \colon (u' \,{}^{f}_{g}\, B)$ there is a unique cell $\lambda \colon (u' \,{}^{A}_{g}\, u)$ such that $\varepsilon' = \lambda | \varepsilon$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u'"', "\bullet" marking] \arrow[dr, phantom, "\varepsilon'"] & B \arrow[d, "1", "\bullet" marking] \\
A' \arrow[r, "g"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "u'"', "\bullet" marking] \arrow[dr, phantom, "\lambda"] & A \arrow[r, "f"] \arrow[d, "u", "\bullet" marking] \arrow[dr, phantom, "\varepsilon"] & B \arrow[d, "1", "\bullet" marking] \\
A' \arrow[r, "g"'] & B \arrow[r, equal] & B
\end{tikzcd}
\tag{52}
$$

In fact, given (η, ε), we can (and must) take $\lambda = \eta \otimes \varepsilon'$; on the other hand, given ε' we define $\eta \colon (A \,{}^{A}_{f}\, u)$ by the equation $\eta | \varepsilon = 1^\bullet_f$ and deduce that $\eta \otimes \varepsilon = 1_u$ because $(\eta \otimes \varepsilon) \,|\, \varepsilon = (\eta | \varepsilon) \otimes \varepsilon = \varepsilon = (1_u | \varepsilon)$.

Similarly, also the pair (u, η) is characterised by a universal property

(b) for every cell $\eta' \colon (A \,{}^{g}_{f}\, u')$ there is a unique cell $\mu \colon (u \,{}^{g}_{B}\, u')$ such that $\eta' = \eta | \mu$.

Therefore, if f has a vertical companion, this is determined up to a unique special isocell, *and will often be written as* f_*. Companions compose in the obvious (covariant) way: if $g \colon B \to C$ also has a companion g_*, then the vertical arrow $g_* f_* \colon A \to\!\!\!\!\!\bullet\;\; C$ is companion to $gf \colon A \to C$, with unit

$$
\left(\frac{\eta \mid 1}{1^\bullet \mid \eta'} \right) \colon \; (A \,{}^{A}_{gf}\, g_* f_*).
$$

Companionship is preserved by *unitary* lax or colax double functors.

We say that $\mathbb{D}$ *has vertical companions* if every horizontal arrow has a vertical companion. The weak double categories recalled in Section 1 have vertical companions, given by the obvious embedding of horizontal arrows into the vertical ones.

Companionship is simpler for horizontal *iso*morphisms. If f is one and has a companion u, then its unit and counit are also horizontally invertible and determine each other:

$$
(\varepsilon \,|\, 1^\bullet_g \,|\, \eta) = \eta \otimes \varepsilon = 1_u \qquad\qquad (g = f^{-1}), \tag{53}
$$

as one can see rewriting $(\varepsilon \,|\, 1^\bullet_g \,|\, \eta)$ as follows, and then applying middle-four interchange

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"', "\bullet" marking] \arrow[dr, phantom, "1^\bullet_f"] & B \arrow[r, "g"] \arrow[d, "1", "\bullet" marking] \arrow[dr, phantom, "1^\bullet_g"] & A \arrow[r, equal] \arrow[d, "1", "\bullet" marking] \arrow[dr, phantom, "\eta"] & A \arrow[d, "u", "\bullet" marking] \\
A \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\varepsilon"] & B \arrow[r, "g"] \arrow[d, "1", "\bullet" marking] \arrow[dr, phantom, "1^\bullet_g"] & A \arrow[r, "f"] \arrow[d, "1", "\bullet" marking] \arrow[dr, phantom, "1^\bullet_f"] & B \arrow[d, "1", "\bullet" marking] \\
B \arrow[r, equal] & B \arrow[r, "g"'] & A \arrow[r, "f"] & B
\end{tikzcd}
$$

Conversely, the existence of a horizontally invertible cell $\eta \colon (A \,{}^{A}_{f}\, u)$ implies that f is horizontally invertible, with companion u and counit as above.

3.2. Orthogonal adjoints. Transforming companionship by horizontal (or vertical) duality, the arrows $f\colon A \to B$ and $v\colon B \to\!\!\!\!\bullet\,\, A$ are made *orthogonal adjoints* by a pair (α, β) of cells as below

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"', "\bullet" marking] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v", "\bullet" marking] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\beta"] & B \arrow[d, "1", "\bullet" marking] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\tag{54}
$$

with $\alpha|\beta = 1^\bullet_f$ and $\beta \otimes \alpha = 1_v$. Then, f is the *horizontal adjoint* and v the *vertical* one. (In the general case there is no reason of distinguishing 'left' and 'right', unit and counit; see the examples of [GP2], Section 1.3).

Again, given f, these relations can be described by universal properties for (v, β) or (v, α)

(a) for every cell $\beta'\colon (v' \,{}^{g}_{f}\, B)$ there is a unique cell $\lambda\colon (v' \,{}^{g}_{A}\, v)$ such that $\beta' = \lambda|\beta$,

(b) for every cell $\alpha'\colon (A \,{}^{f}_{g}\, v')$ there is a unique cell $\mu\colon (v \,{}^{B}_{g}\, v')$ such that $\alpha' = \alpha|\mu$.

The vertical adjoint of f is determined up to a special isocell and will often be written as f^*; vertical adjoints compose, contravariantly: $(gf)^*$ can be constructed as f^*g^*.

We say that $\mathbb{D}$ *has vertical adjoints* if every horizontal arrow has a vertical adjoint. Plainly, this is the case for the weak double categories recalled in Section 1.

3.3. Proposition. *Let $f\colon A \to B$ have a vertical companion $u\colon A \to\!\!\!\!\bullet\,\, B$. Then the arrow $v\colon B \to\!\!\!\!\bullet\,\, A$ is vertical adjoint to f if and only if $u \dashv v$ in the bicategory $\mathbf{V}\mathbb{D}$ (of objects, vertical arrows and special cells).*

Proof. Given four cells $\eta, \varepsilon, \alpha, \beta$ as above (in 3.1, 3.2), we have two special cells

$$\eta \otimes \alpha\colon 1^\bullet \to u \otimes v, \qquad\qquad \beta \otimes \varepsilon\colon u \otimes v \to 1^\bullet,$$

that are easily seen to satisfy the triangle identities in $\mathbf{V}\mathbb{D}$. The converse is similarly obvious. ■

4. Multiple adjunctions

We now define a colax/lax adjunction between chiral multiple categories, a notion that occurs naturally in various situations, as already seen in Section 1.

4.1. Colax/lax adjunctions. A *colax/lax cm-adjunction* $(\eta, \varepsilon)\colon F \dashv G$, or a colax/lax adjunction between chiral multiple categories, will be an orthogonal adjunction *in* the double category $\mathbb{C}\mathrm{mc}$ (as defined in 3.2).

The data amount thus to:

- a *colax* cm-functor $F\colon \mathsf{X} \to \mathsf{A}$, the left adjoint,

- a *lax* cm-functor $G\colon \mathsf{A} \to \mathsf{X}$, the right adjoint,

- two $\mathbb{C}$mc-cells $\eta\colon 1_{\mathsf{X}} \dashrightarrow GF$ and $\varepsilon\colon FG \dashrightarrow 1_{\mathsf{A}}$ (unit and counit) that satisfy the triangle equalities:

$$\begin{tikzcd} \mathsf{X} \arrow[r, equal] \arrow[d, "F\bullet"'] \arrow[dr, phantom, "\eta"] & \mathsf{X} \arrow[d, equal] \\ \mathsf{A} \arrow[r, "G"'] & \mathsf{X} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{A} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \mathsf{X} \arrow[d, "\bullet F"] \\ \mathsf{A} \arrow[r, equal] & \mathsf{A} \end{tikzcd} \qquad \eta \otimes \varepsilon = 1_F, \quad \varepsilon \,|\, \eta = 1^\bullet_G. \tag{55}$$

We speak of a *pseudo/lax* (resp. a *colax/pseudo*) adjunction when the left (resp. right) adjoint is pseudo, and of a *pseudo* (or *strict*) adjunction when both adjoints are pseudo (or strict).

From general properties (see 3.2), we already know that the left adjoint of a lax cm-functor G is determined up to transversal isomorphism (which amounts to a special invertible cell between vertical arrows in $\mathbb{C}$mc) and that left adjoints compose, contravariantly. Similarly for right adjoints.

As in 2.2, the arrow of a colax cm-functor is marked with a dot *when displayed vertically*, in a double cell of $\mathbb{C}$mc. Again, we may write unit and counit as $\eta\colon 1 \dashrightarrow GF$ and $\varepsilon\colon FG \dashrightarrow 1$, *but we should recall that the coherence conditions of such 'transformations' involve the comparison cells of F and G.* Therefore (as with double categories, in [GP2]), a *general* colax/lax *adjunction cannot be seen as an adjunction in some bicategory*; although this is possible for a *pseudo/lax* or a *colax/pseudo* adjunction, as we shall prove in the next section.

4.2. A description. To make the previous definition explicit, a *colax/lax adjunction* $(\eta, \varepsilon)\colon\ F \dashv G$ between the cm-categories X and A consists of the following items.

(a) A *colax* cm-functor $F\colon \mathsf{X} \to \mathsf{A}$, with comparison transversal maps

$$\underline{F}_i(x)\colon F(e_i x) \to_0 e_i(Fx), \qquad\qquad \underline{F}_i(x,y)\colon F(x +_i y) \to_0 Fx +_i Fy.$$

(b) A *lax* cm-functor $G\colon \mathsf{A} \to \mathsf{X}$, with comparison transversal maps

$$\underline{G}_i(a)\colon e_i(Ga) \to_0 G(e_i a), \qquad\qquad \underline{G}_i(a,b)\colon Ga +_i Gb \to_0 G(a +_i b).$$

(c) An ordinary adjunction $F_{\mathbf{i}} \dashv G_{\mathbf{i}}$ for every positive multi-index $\mathbf{i}$

$$\eta_{\mathbf{i}}\colon 1 \to G_{\mathbf{i}}F_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{X}), \qquad \varepsilon_{\mathbf{i}}\colon F_{\mathbf{i}}G_{\mathbf{i}} \to 1\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{A}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A}),$$
$$\varepsilon_{\mathbf{i}}F_{\mathbf{i}}.F_{\mathbf{i}}\eta_{\mathbf{i}} = 1_{F_{\mathbf{i}}}, \qquad G_{\mathbf{i}}\varepsilon_{\mathbf{i}}.\eta_{\mathbf{i}}G_{\mathbf{i}} = 1_{G_{\mathbf{i}}}.$$

(Note that *there is an abuse of notation*: for the sake of simplicity we write as $F_{\mathbf{i}}$ the functor $\mathrm{tv}_{\mathbf{i}}(F)\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$, which involves the components $F_{\mathbf{i}}\colon X_{\mathbf{i}} \to A_{\mathbf{i}}$ and $F_{0\mathbf{i}}\colon X_{0\mathbf{i}} \to A_{0\mathbf{i}}$ of F.)

Explicitly this means that we are assigning:

- transversal maps $\eta_{\mathbf{i}}x\colon x \to_0 G_{\mathbf{i}}F_{\mathbf{i}}x$ in X (for x in $X_{\mathbf{i}}$), also written as $\eta x\colon x \to_0 GFx$,
- transversal maps $\varepsilon_{\mathbf{i}}a\colon F_{\mathbf{i}}G_{\mathbf{i}}a \to_0 a$ in A (for a in $A_{\mathbf{i}}$), also written as $\varepsilon a\colon FGa \to_0 a$,

satisfying the naturality conditions (c1) and the triangle identities (c2), for transversal maps $f\colon x \to_0 y$ in X and $h\colon a \to_0 b$ in A

(c1) $\eta y.f = GFf.\eta x$, $\qquad\qquad \varepsilon b.FGh = h.\varepsilon a$,

(c2) $\varepsilon Fx.F\eta x = 1_{Fx}$, $\qquad\qquad G\varepsilon a.\eta Ga = 1_{Ga}$.

(d) These families $\eta = (\eta_{\mathbf{i}})$ and $\varepsilon = (\varepsilon_{\mathbf{i}})$ must respect faces and be coherent with the positive compositions (in terms of the comparison cells of F and G):

$$\eta(\partial_i^\alpha x) = \partial_i^\alpha(\eta x), \qquad\qquad \varepsilon(\partial_i^\alpha a) = \partial_i^\alpha(\varepsilon a), \tag{56}$$

(d1) (*coherence of* η and ε *with i-identities*) for x in X and a in A:

$$G\underline{F}_i(x).\eta(e_ix) = \underline{G}_i(Fx).e_i(\eta x) \qquad (\eta(e_i(x) = e_i(\eta x), \text{ if } F \text{ and } G \text{ are unitary}), \tag{57}$$

$$\varepsilon(e_ia).F\underline{G}_i(a) = e_i(\varepsilon a).\underline{F}_i(Ga) \qquad (\varepsilon(e_ia) = e_i(\varepsilon a), \text{ if } F \text{ and } G \text{ are unitary}), \tag{58}$$

(d2) (*coherence of* η and ε *with i-composition*) for $z = x +_i y$ in X and $c = a +_1 b$ in A:

$$G\underline{F}_i(x,y).\eta z = \underline{G}_i(Fx,Fy).(\eta x +_i \eta y), \tag{59}$$

$$\varepsilon c.F\underline{G}_i(a,b) = (\varepsilon a +_i \varepsilon b).\underline{F}_i(Ga,Gb). \tag{60}$$

$$\begin{tikzcd}
z \arrow[r, "\eta z"] \arrow[d, "\eta x +_i \eta y"'] & GFz \arrow[d, "G\underline{F}_i(x,y)"] \\
GFx +_i GFy \arrow[r, "\underline{G}_i(Fx,Fy)"'] & G(Fx +_i Fy)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(Ga + Gb) \arrow[r, "F\underline{G}_i(a,b)"] \arrow[d, "\underline{F}_i(Ga,Gb)"'] & FGc \arrow[d, "\varepsilon c"] \\
FGa + FGb \arrow[r, "\varepsilon a +_i \varepsilon b"'] & c
\end{tikzcd}$$

4.3. Lemma. *(a) In a colax/lax cm-adjunction $(\eta, \varepsilon)\colon F \dashv G$, the comparison maps of G determine the comparison maps of F, through the ordinary adjunctions $F_{\mathbf{i}} \dashv G_{\mathbf{i}}$, as*

$$\begin{gathered}\underline{F}_i(x) = \varepsilon e_i(Fx).F\underline{G}_i(Fx).Fe_i(\eta x)\colon \\ Fe_i(x) \to Fe_i(GFx) \to FG(e_iFx) \to e_iFx,\end{gathered} \tag{61}$$

$$\begin{gathered}\underline{F}_i(x,y) = \varepsilon(Fx +_i Fy).F\underline{G}_i(Fx,Fy).F(\eta x +_i \eta y)\colon \\ F(x +_i y) \to F(GFx +_i GFy) \to FG(Fx +_i Fy) \to Fx +_i Fy.\end{gathered} \tag{62}$$

Dually, the comparison maps of F determine the comparison maps of G, through the ordinary adjunctions.

(b) If all the components of η, ε are invertible, then G is pseudo if and only if F is.

Note. Loosely speaking, point (a) says that a lax multiple functor *can only have* a colax left adjoint (if any), and symmetrically. This fact will be completed in Theorem 5.3, showing that if a lax functor has a *lax* adjoint, the latter is necessarily pseudo.

Proof. (a) The first equation of (d1) says that the adjoint map of $\underline{F}_i(x)$, i.e. $(\underline{F}_i(x))' = G\underline{F}_i(x).\eta(e_ix)$, must be equal to $f = \underline{G}_i(Fx).e_i(\eta x)$. The adjoint map of the latter gives $\underline{F}_i(x) = f' = \varepsilon e_i(Fx).F(f)$.

In the same way the first equation of (d2) determines $\underline{F}_i(x,y)$. Point (b) is a straightforward consequence. ∎

4.4. Theorem. [Characterisation by transversal hom-sets] *A multiple adjunction $(\eta,\varepsilon)\colon F \dashv G$ can equivalently be given by a colax cm-functor $F\colon \mathsf{X} \to \mathsf{A}$, a lax cm-functor $G\colon \mathsf{A} \to \mathsf{X}$ and a family $(L_\mathbf{i})$ of functorial isomorphisms indexed by the positive multi-indices $\mathbf{i} \subset \mathbb{N}$*

$$\begin{gathered} L_\mathbf{i}\colon\ \mathrm{tv}_\mathbf{i}(\mathsf{A})(F_\mathbf{i}(-),=)\ \to\ \mathrm{tv}_\mathbf{i}(\mathsf{X})(-,G_\mathbf{i}(=))\colon \mathrm{tv}_\mathbf{i}(\mathsf{X})^{\mathrm{op}}\times \mathrm{tv}_\mathbf{i}(\mathsf{A}) \to \mathbf{Set}, \\ L_\mathbf{i}(x,a)\colon\ \mathrm{tv}_\mathbf{i}(\mathsf{A})(Fx,a) \to \mathrm{tv}_\mathbf{i}(\mathsf{X})(x,Ga). \end{gathered} \tag{63}$$

The components $L_\mathbf{i}(x,a)$, also written as $L(x,a)$ or just L, have to commute with faces and be coherent with the positive operations (through the comparison maps of F and G), i.e. must satisfy the following conditions (ad.1-3):

(ad.1) $L_\mathbf{i}(\partial_i^\alpha x, \partial_i^\alpha a) = \partial_i^\alpha(L_\mathbf{i}(x,a))$,

(ad.2) $L(e_ix, e_ia)(e_i(h).\underline{F}(x)) = \underline{G}(a).e_i(Lh)$ (*for* $h\colon Fx \to_0 a$ *in* $\mathrm{tv}_{\mathbf{i}|i}(\mathsf{A})$),

$$Fe_i(x) \xrightarrow{\underline{F}(x)} e_i(Fx) \xrightarrow{e_i(h)} e_i(a) \qquad\qquad e_i(x) \xrightarrow{e_i(Lh)} e_i(Ga) \xrightarrow{\underline{G}(a)} Ge_i(a)$$

(ad.3) $L((h+_i k).\underline{F}_i(x,y)) = \underline{G}_i(a,b).(Lh +_i Lk)$ ($h\colon Fx \to_0 a,\ k\colon Fy \to_0 b$ *in* $\mathrm{tv}_\mathbf{i}(\mathsf{A})$),

$$F(x+_i y) \xrightarrow{\underline{F}_i(x,y)} Fx +_i Fy \xrightarrow{h+ik} a +_i b$$

$$x +_i y \xrightarrow{Lh+_iLk} Ga +_i Gb \xrightarrow{\underline{G}_i(a,b)} G(a +_i b)$$

In this equivalence, $L_\mathbf{i}(x,a)$ is defined by the unit η as

$$L_\mathbf{i}(x,a)(h) = Gh.\eta_\mathbf{i}x\colon x \to_0 GFx \to_0 Ga \qquad (\textit{for } h\colon Fx \to_0 a \textit{ in } \mathrm{tv}_\mathbf{i}(\mathsf{A})). \tag{64}$$

The other way round, the component $\eta_\mathbf{i}\colon 1 \to G_\mathbf{i}F_\mathbf{i}\colon X_\mathbf{i} \to X_\mathbf{i}$ of the unit is defined by L as

$$\eta_\mathbf{i}(x) = L_\mathbf{i}(x,Fx)(\mathrm{id}Fx)\colon x \to_0 GF(x) \qquad (\textit{for } x \textit{ in } X_\mathbf{i}). \tag{65}$$

Proof. We have only to verify the equivalence of the conditions (56)-(60) with the conditions above.

This is straightforward. For instance, to show that (59) implies (ad.3), let $h\colon Fx \to a$ and $k\colon Fy \to b$ be i-consecutive $\mathbf{i}$-maps in A, and apply $L = L(x+_i y, a+_i a)$ as defined above, in (64):

$$\begin{aligned} L((h+_i k).\underline{F}_i(x,y)) &= G(h+_i k).G\underline{F}_i(x,y).\eta(x+_i y) \\ &= G(h+_i k).\underline{G}_i(Fx,Fy).(\eta x+_i \eta y) && \text{(by (59))} \\ &= \underline{G}_i(a,b).(Gh+_i Gk).(\eta x+_i \eta y) && \text{(by 2.1, axiom (lmf.2))} \\ &= \underline{G}_i(a,b).(Lh+_i Lk). \end{aligned}$$

■

4.5. Corollary. [Characterisation by commas] *With the previous notation, a multiple adjunction amounts to an isomorphism of chiral multiple categories* $L\colon F\downarrow\mathsf{A} \to \mathsf{X}\downarrow G$ *over the cartesian product* $\mathsf{X}\times\mathsf{A}$

$$\begin{tikzcd} F\downarrow\mathsf{A} \arrow[rr, "L"] \arrow[dr] & {=} & \mathsf{X}\downarrow G \arrow[dl] \\ & \mathsf{X}\times\mathsf{A} & \end{tikzcd} \tag{66}$$

Proof. This is a straightforward consequence of the previous theorem. ■

4.6. Theorem. [Right adjoint by universal properties] *Let a colax cm-functor* $F\colon \mathsf{X}\to\mathsf{A}$ *be given.*

*The existence (*and choice*) of a right adjoint lax cm-functor* G *amounts to a family (rad.*$\mathbf{i}$*) of conditions and choices, indexed by the positive multi-indices* $\mathbf{i}$*:*

(rad.$\mathbf{i}$) *for every* $\mathbf{i}$*-cube* a *in* A *there is a universal arrow* $(Ga, \varepsilon_{\mathbf{i}}a\colon F(Ga)\to_0 a)$ *from the functor* $F_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X})\to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ *to the object* x*,* and we choose one*,*

provided that these choices commute with faces.

Explicitly, the universal property means that, for each $\mathbf{i}$*-cube* x *in* X *and* $\mathbf{i}$*-map* $h\colon Fx\to_0 a$ *in* A *there is a unique* $f\colon x\to_0 Ga$ *such that* $h=\varepsilon a.Ff\colon Fx\to_0 F(Ga)\to_0 a$*.*

The comparison $\mathbf{i}$*-maps of* G

$$\underline{G}_i(a)\colon e_i(Ga)\to_0 G(e_ia), \qquad \underline{G}_i(a,b)\colon Ga+_i Gb\to_0 G(a+_i b), \tag{67}$$

are then given by the universal property of ε*, as the unique solution of the equations* (58), (60)*, respectively.*

Proof. The conditions (rad.$\mathbf{i}$) are plainly necessary, including consistency with faces.

Conversely, each (rad.$\mathbf{i}$) provides an ordinary adjunction $(\eta_{\mathbf{i}},\varepsilon_{\mathbf{i}})\colon F_{\mathbf{i}}\dashv G_{\mathbf{i}}$ for the categories $\mathrm{tv}_{\mathbf{i}}(\mathsf{X})$, $\mathrm{tv}_{\mathbf{i}}(\mathsf{A})$, so that G, η and ε are correctly defined - as far as cubes, transversal maps, faces, transversal composition and transversal identities are concerned.

Now we define the comparison maps $\underline{G}_i$ as specified in the statement, so that the coherence properties of ε are satisfied (see (58), (60)). One verifies easily, for such transversal maps, the axioms of naturality and coherence (see 2.1).

Finally, we have to prove that $\eta\colon 1 \dashrightarrow GF$ satisfies the coherence property (59)

$$G\underline{F}_i(x,y).\eta z = \underline{G}_i(Fx,Fy).(\eta x+_i \eta y), \tag{68}$$

with respect to a composition $z = x +_i y$ of **i**-cubes in X (similarly one proves (57)). By the universal property of ε, it will suffice to show that the composite $\varepsilon(Fx +_i Fy).F(-)$ takes the same value on both terms of (68). In fact, on the left-hand term we get $\underline{F}_i(x, y)$

$$\varepsilon(Fx +_i Fy).FG\underline{F}_i(x, y).F\eta z = \underline{F}_i(x, y).\varepsilon Fz.F\eta z = \underline{F}_i(x, y).$$

We get the same on the right-hand term of (68), using (60), the naturality of $\underline{F}_i$, the $0i$-interchange in A and a triangle identity

$$\begin{aligned}
&\varepsilon(Fx +_i Fy).F\underline{G}_i(Fx, Fy).F(\eta x +_i \eta y) \\
&\quad = (\varepsilon Fx +_i \varepsilon Fy).\underline{F}_i(GFx, GFy).F(\eta x +_i \eta y) \\
&\quad = (\varepsilon Fx +_i \varepsilon Fy).(F\eta x +_i F\eta y).\underline{F}_i(x, y) \\
&\quad = (\varepsilon Fx.F\eta x +_i \varepsilon Fy.F\eta y).\underline{F}_i(x, y) = (1_{Fx} +_i 1_{Fy}).\underline{F}_i(x, y) = \underline{F}_i(x, y).
\end{aligned}$$

■

4.7. Theorem. [Factorisation of adjunctions] *Let $F \dashv G$ be a colax/lax adjunction between X and A. Then, using the isomorphism of cm-categories $L\colon F{\downarrow}\mathsf{A} \to \mathsf{X}{\downarrow}G$ (Corollary 4.5), we can factorise the adjunction as a composite of:*

$$\mathsf{X} \underset{P}{\overset{F'}{\rightleftarrows}} F{\downarrow}\mathsf{A} \underset{L^{-1}}{\overset{L}{\rightleftarrows}} \mathsf{X}{\downarrow}G \underset{G'}{\overset{Q}{\rightleftarrows}} \mathsf{A} \qquad F = QLF', \quad G = PL^{-1}G'. \tag{69}$$

- *a* coreflective *colax/strict adjunction $F' \dashv P$ (with unit $PF' = 1$),*
- *an isomorphism $L \dashv L^{-1}$,*
- *a* reflective *strict/lax adjunction $Q \dashv G'$ (with counit $QG' = 1$),*

where the comma projections P and Q are strict cm-functors.

Proof. We define the lax cm-functor $G'\colon \mathsf{A} \to \mathsf{X} \downarrow G$ by the universal property of commas 2.6(a), applied to $G\colon \mathsf{A} \to \mathsf{X}$, $1\colon \mathsf{A} \to \mathsf{A}$ and $\varphi = 1^{\bullet}_G$ as in the diagram below

$$\begin{tikzcd}
\mathsf{A} \arrow[rr, "G"] \arrow[d, "1"'] & \arrow[d, phantom, "\varphi"] & \mathsf{X} \arrow[d, "1"] \\
\mathsf{A} \arrow[r, "1"'] & \mathsf{A} \arrow[r, "G"'] & \mathsf{X}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathsf{A} \arrow[r, "G'"] \arrow[d, "1"'] \arrow[dr, phantom, "\psi"] & 1{\downarrow}G \arrow[r, "P"] \arrow[d, "Q"] \arrow[dr, phantom, "\pi"] & \mathsf{X} \arrow[d, "1"] \\
\mathsf{A} \arrow[r, "1"'] & \mathsf{A} \arrow[r, "G"'] & \mathsf{X}
\end{tikzcd} \tag{70}$$

$$\begin{gathered}
G'(a) = (Ga, a; 1\colon Ga \to Ga), \\
\underline{G}'_i(a) = (\underline{G}_i(a), 1)\colon (e_iGa, e_ia; \underline{G}_i(a)) \to (G(e_ia), e_ia; 1), \\
\underline{G}'_i(a, b) = (\underline{G}_i(a, b), 1)\colon (Ga +_i Gb, a +_i b; \underline{G}_i(a, b)) \to (G(a +_i b), a +_i b; 1).
\end{gathered}$$

Similarly, we define the colax cm-functor $F'\colon \mathsf{X} \to F{\downarrow}\mathsf{A}$ by the dual result (2.6(b))

$$\begin{gathered}
F'(x) = (x, Fx; 1\colon Fx \to Fx), \\
\underline{F}'_i(x) = (1, \underline{F}_i(x))\colon (e_ix, F(e_ix); 1) \to (e_ix, Fe_ix; \underline{F}_i(x)), \\
\underline{F}'_i(x, y) = (1, \underline{F}_i(x, y))\colon (x +_i y, F(x +_i y); 1) \to (x +_i y, Fx +_i Fy; \underline{F}_i(x, y)).
\end{gathered} \tag{71}$$

The coreflective adjunction $F' \dashv P$ is obvious

$$\begin{gathered} \eta' x = 1_x\colon x \to PF'x,\\ \varepsilon'(x,a;f\colon Fx\to a) = (1_x,f)\colon (x,Fx;1\colon Fx\to Fx)\to(x,a;f\colon Fx\to a), \end{gathered} \qquad (72)$$

as well as the reflective adjunction $Q \dashv G'$ and the factorisation above. ■

5. Multiple adjunctions and pseudo cm-functors

We consider now cm-adjunctions where the left or right adjoint is a pseudo cm-functor. Then we introduce adjoint equivalences of chiral multiple categories.

5.1. Comments. Let us recall, from 4.1, that a *pseudo/lax* cm-adjunction $F \dashv G$ is a colax/lax adjunction between cm-categories where the left adjoint F is pseudo.

Then the comparison cells of F are horizontally invertible and the composites GF and FG are lax cm-functors; it follows (from definition 2.2) that the unit and counit are horizontal transformations of such functors. Therefore *a pseudo/lax cm-adjunction gives an adjunction in the 2-category* Lx**Cmc** of cm-categories, lax cm-functors and transversal transformations (see 2.2); and we shall prove that these two facts are actually equivalent (Theorem 5.3).

Dually a *colax/pseudo* cm-adjunction, where the right adjoint G is pseudo, will amount to an adjunction in the 2-category Cx**Cmc** of cm-categories, colax cm-functors and transversal transformations. Finally a *pseudo* cm-adjunction, where both F and G are pseudo, will be the same as an adjunction in the 2-category Ps**Cmc** whose arrows are the pseudo cm-functors.

5.2. Theorem. [Companions in $\mathbb{C}$mc] *A lax cm-functor G has an orthogonal companion F in* the double category $\mathbb{C}$mc *if and only if it is pseudo; then one can define $F = G_*$ as the colax cm-functor which coincides with G except for comparison maps, that are transversally inverse to those of G.*

Proof. We restrict to unitary cm-categories, for simplicity. If G is pseudo, it is obvious that G_*, as defined above, is an orthogonal companion.

Conversely, suppose that $G\colon \mathsf{A} \to \mathsf{X}$ (lax) has an orthogonal companion F (colax). There are thus two double cells η, ε in $\mathbb{C}$mc)

$$\begin{tikzcd} \mathsf{A} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\eta"] & \mathsf{A} \arrow[d, "F", "\bullet"'] \\ \mathsf{A} \arrow[r, "G"'] & \mathsf{X} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{A} \arrow[r, "G"] \arrow[d, "F"', "\bullet"] \arrow[dr, phantom, "\varepsilon"] & \mathsf{X} \arrow[d, equal] \\ \mathsf{X} \arrow[r, equal] & \mathsf{X} \end{tikzcd} \qquad (73)$$

which satisfy the identities $\eta|\varepsilon = 1^\bullet_G$, $\eta \otimes \varepsilon = 1_F$.

This means two 'transformations' $\eta\colon F \dashrightarrow G$, $\varepsilon\colon G \dashrightarrow F$, as defined in 2.2; for every **i**-cube a in A, we have two transversal maps ηa and εa in X

$$\eta a\colon Fa \to Ga, \qquad\qquad \varepsilon a\colon Ga \to Fa, \tag{74}$$

consistently with faces. These maps are transversally inverse, because of the previous identities (see (36))

$$\eta a.\varepsilon a = (\eta \,|\, \varepsilon)(a) = 1_{Ga}, \qquad \varepsilon a.\eta a = (\eta \otimes \varepsilon)(a) = 1_{Fa}. \tag{75}$$

Applying now the coherence condition (dc.3) (of 2.2) for the transformations η, ε and a concatenation $c = a +_i b$ in A we find

$$\begin{gathered} \eta c = \underline{G}_i(a,b).(\eta a +_i \eta b).\underline{F}_i(a,b)\colon Fc \to Gc, \\ \varepsilon a +_i \varepsilon b = \underline{F}_i(a,b).\varepsilon c.\underline{G}_i(a,b)\colon Ga +_i Gb \to Fa +_i Fb. \end{gathered} \tag{76}$$

Since all the components of η and ε are transversally invertible, this proves that $\underline{G}_i(a,b)$ has a left inverse and a right inverse transversal map, whence it is invertible. Similarly for degeneracies.

Therefore G is pseudo and F is transversally isomorphic to G_*. ■

5.3. THEOREM. (a) (Pseudo/lax adjunctions) *For every adjunction $F \dashv G$ in the 2-category* Lx**Cmc**, *the functor F is pseudo and the adjunction is pseudo/lax, in the sense of 4.1 (or 5.1).*

(b) (Colax/pseudo adjunctions) *For every adjunction $F \dashv G$ in the 2-category* Cx**Cmc**, *the functor G is pseudo and the adjunction is colax/pseudo, in the sense of 4.1 (or 5.1).*

More formally, (a) can be rewritten saying that, in the double category $\mathbb{C}$mc, *if the horizontal arrow G has a 'horizontal left adjoint' F (within the horizontal 2-category* **H**$\mathbb{C}$mc = Lx**Cmc**), *then it also has an orthogonal adjoint G^* (colax). (Then, applying Proposition 3.3, it would follow that F and G^* are companions, whence F is pseudo, by Theorem 5.2, and isomorphic to G^*.)*

PROOF. It suffices to prove (a); again, we only deal with the comparisons of a composition.

Let the lax structures of $F\colon \mathsf{X} \to \mathsf{A}$ and $G\colon \mathsf{A} \to \mathsf{X}$ be given by the following comparison maps, where $z = x +_i y$ and $c = a +_i b$

$$\lambda_i(x,y)\colon Fx +_i Fy \to F(x +_i y), \qquad \underline{G}_i(a,b)\colon Ga +_i Gb \to G(a +_i b),$$

so that we have:

$$\begin{aligned} \eta z &= G\lambda_i(x,y).\underline{G}_i(Fx,Fy).(\eta x +_i \eta y)\colon \\ &\qquad z \to GFx +_i GFy \to G(Fx +_i Fy) \to GFz, \\ \varepsilon a +_i \varepsilon b &= \varepsilon c.F\underline{G}_i(a,b).\lambda_i(Ga,Gb)\colon \\ &\qquad FGa +_i FGb \to F(Ga +_i Gb) \to FG(a +_i b) \to c. \end{aligned} \tag{77}$$

We construct a colax structure $\underline{F}$ for F, letting

$$\underline{F}_i(x,y) = \varepsilon(Fx +_i Fy).F\underline{G}_i(Fx,Fy).F(\eta x +_i \eta y)\colon$$
$$Fz \to F(GFx +_i GFy) \to FG(Fx +_i Fy) \to Fx +_i Fy.$$

Now it is sufficient to verify that $\underline{F}_i(x,y)$ and $\lambda_i(x,y)$ are transversally inverse:

$$\begin{aligned}
&\lambda_i(x,y).\underline{F}_i(x,y)\\
&\quad = \lambda_i(x,y).\varepsilon(Fx +_i Fy).F\underline{G}_i(Fx,Fy).F(\eta x +_i \eta y)\\
&\quad = \varepsilon Fz.FG\lambda_i(x,y).F\underline{G}_i(Fx,Fy).F(\eta x +_i \eta y) && \text{(by naturality of } \varepsilon\text{, see 4.2),}\\
&\quad = \varepsilon F(z).F(\eta z) = 1_{Fz} && \text{(by (77),}
\end{aligned}$$

$$\begin{aligned}
&\underline{F}_i(x,y).\lambda_i(x,y)\\
&\quad = \varepsilon(Fx +_i Fy).F\underline{G}_i(Fx,Fy).F(\eta x +_i \eta y).\lambda_i(x,y)\\
&\quad = \varepsilon(Fx +_i Fy).F\underline{G}_i(Fx,Fy).\lambda_i(GFx,GFy).(F\eta x +_i F\eta y) && \text{(by naturality of } \lambda\text{),}\\
&\quad = (\varepsilon Fx +_i \varepsilon Fy).(F\eta x +_i F\eta y) && \text{(by (77)),}\\
&\quad = \varepsilon Fx.F\eta x +_i \varepsilon Fy.F\eta y = 1_{Fx} +_i 1_{Fy} = 1_{Fx+iFy} && \text{(by } 0i\text{-interchange).}
\end{aligned}$$

■

5.4. Equivalences of cm-categories. An *adjoint equivalence* between two cm-categories X and A will be a pseudo cm-adjunction $(\eta,\varepsilon)\colon F \dashv G$ where the transversal transformations $\eta\colon 1_{\mathsf{X}} \to GF$ and $\varepsilon\colon FG \to 1_{\mathsf{A}}$ are invertible.

The following properties of a cm-functor $F\colon \mathsf{X} \to \mathsf{A}$ will allow us (in the next theorem) to characterise this fact in the usual way, under the mild restriction of *transversal invariance* (see II.1.6):

(a) We say that F is *faithful* if all the ordinary functors $F_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ (between the categories of $\mathbf{i}$-cubes and their transversal maps) are faithful: given two $\mathbf{i}$-maps $f,g\colon x \to_0 y$ of X between the same $\mathbf{i}$-cubes, $F(f) = F(g)$ implies $f = g$.

(b) Similarly, we say that F is *full* if all the ordinary functors $F_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ are: for every $\mathbf{i}$-map $h\colon F(x) \to_0 F(y)$ in A there is an $\mathbf{i}$-map $f\colon x \to_0 y$ in X such that $F(f) = h$.

(c) Finally, we say that F is *essentially surjective on cubes* if every $F_{\mathbf{i}}$ is: for every $\mathbf{i}$-cube a in A there is some $\mathbf{i}$-cube x in X and some invertible $\mathbf{i}$-map $h\colon F(x) \to_0 a$ in A.

5.5. Theorem. [Characterisations of equivalences] *Let $F\colon \mathsf{X} \to \mathsf{A}$ be a pseudo cm-functor between two* transversally invariant *cm-categories (see II.1.6). The following conditions are equivalent:*

(i) $F\colon \mathsf{X} \to \mathsf{A}$ belongs to an adjoint equivalence of cm-categories,

(ii) F is faithful, full and essentially surjective on cubes (see 5.4),

(iii) every ordinary functor $F_{\mathbf{i}}\colon \mathrm{tv}_{\mathbf{i}}(\mathsf{X}) \to \mathrm{tv}_{\mathbf{i}}(\mathsf{A})$ is an equivalence of categories.

Moreover, if F is unitary, one can make its 'quasi-inverse' unitary as well.

Remark. *The axiom of choice is assumed.*

Proof. By our previous definitions in 5.4, conditions (ii) and (iii) are about the family of ordinary functors ($F_{\mathbf{i}}$) and are well known to be equivalent (assuming (AC)). Moreover, if F belongs to an adjoint equivalence $(\eta, \varepsilon)\colon F \dashv G$, every $F_{\mathbf{i}}$ is obviously an equivalence of categories.

Conversely, let us assume that every $F_{\mathbf{i}}$ is an equivalence of ordinary categories and let us extend the pseudo cm-functor F to an adjoint equivalence, proceeding by induction on the degree $n \geqslant 0$ of the positive multi-index $\mathbf{i}$.

First, $F_*\colon \mathrm{tv}_*(\mathsf{X}) \to \mathrm{tv}_*(\mathsf{A})$ is an equivalence of categories and we begin by choosing an adjoint quasi-inverse $G_*\colon \mathrm{tv}_*(\mathsf{A}) \to \mathrm{tv}_*(\mathsf{X})$.

In other words, we choose for every $\star$-cube (or object) a some $G(a)$ in X and some isomorphism $\varepsilon a\colon FG(a) \to a$ in A; then a transversal map $h\colon a \to b$ in A is sent to the unique X-map $G(h)\colon G(a) \to G(b)$ coherent with the previous choices (since F_* is full and faithful). Finally the isomorphism $\eta x\colon x \to GF(x)$ is determined by the triangle equations (for every $\star$-cube x of X).

Assume now that the components of G, ε and η have been defined up to degree $n-1 \geqslant 0$, and let us define them for a multi-index $\mathbf{i}$ of degree n, taking care that the new choices be consistent with the previous ones.

First, for every $\mathbf{i}$-cube a in A we want to choose some $\mathbf{i}$-cube $G(a)$ in X and some $\mathbf{i}$-isomorphism $\varepsilon a\colon FG(a) \to a$ in A, consistently with all faces ∂_i^α ($i \in \mathbf{i}$). In fact there exists (and we choose) some $\mathbf{i}$-cube x and some $\mathbf{i}$-isomorphism $u\colon F(x) \to a$. Then, by the inductive hypothesis, we have a family of $2n$ transversal isomorphisms of A

$$v_i^\alpha = \partial_i^\alpha u^{-1}.\varepsilon(\partial_i^\alpha a)\colon FG(\partial_i^\alpha a) \to \partial_i^\alpha a \to F(\partial_i^\alpha x) \qquad (i \in \mathbf{i}, \alpha = \pm),$$

which can be uniquely lifted as transversal isomorphisms t_i^α of X, since F is full and faithful

$$t_i^\alpha\colon G(\partial_i^\alpha a) \to \partial_i^\alpha x, \qquad\qquad v_i^\alpha = F(t_i^\alpha).$$

The family (v_i^α) has consistent positive faces (see II.1.6), because this is true of the family $(\partial_i^\alpha u^{-1})_{i,\alpha}$, by commuting faces, and of the family $(\varepsilon(\partial_i^\alpha a))_{i,\alpha}$ by inductive assumption. It follows that also the family (t_i^α) has consistent positive faces.

By transversal invariance in X we can fill this family (t_i^α) with a (chosen) transversal $\mathbf{i}$-isomorphism $t\colon y \to x$, and we define the $\mathbf{i}$-cube G(a) and the $\mathbf{i}$-isomorphism εa as follows:

$$G(a) = y, \qquad\qquad \varepsilon a = u.Ft\colon FG(a) \to F(x) \to a.$$

This choice is consistent with faces: $\partial_i^\alpha(\varepsilon a) = (\partial_i^\alpha u).Ft_i^\alpha = (\partial_i^\alpha u).v_i^\alpha = \varepsilon(\partial_i^\alpha a)$.

Now, since $F_{\mathbf{i}}$ is full and faithful, a transversal $\mathbf{i}$-map $h\colon a \to b$ in A is sent to the unique X-map $G(h)\colon G(a) \to G(b)$ satisfying the condition $\varepsilon b.F(Gh) = h.\varepsilon a$ (naturality of ε).

Again, the $\mathbf{i}$-isomorphism $\eta x\colon x \to GF(x)$ is determined by the triangle equations, for every $\mathbf{i}$-cube x of X.

The comparison $\mathbf{i}$-maps $\underline{G}_i$ are uniquely determined by their coherence conditions (see 4.2), for an $\mathbf{i}|i$-cube a and an i-composition of $\mathbf{i}$-cubes $c = a +_i b$ in A

$$\varepsilon e_i a.F\underline{G}_i(a) = e_i(\varepsilon a).\underline{F}_i(Ga), \qquad \varepsilon c.F\underline{G}_i(a,b) = (\varepsilon a +_i \varepsilon b).\underline{F}_i(Ga, Gb).$$

Moreover $\underline{G}_i(a)$ and $\underline{G}_i(a,b)$ are invertible, because so are their images by F, full and faithful.

The construction of G, ε and η is now achieved. One ends by proving that G is indeed a pseudo cm-functor, and that ε, η are coherent with the comparison cells of F and G.

Finally, let us assume that F is unitary: $\underline{F}_i(x)\colon F(e_i x) \to e_i(Fx)$ is always an identity. To make G unitary we assume that - in the previous inductive construction - the following constraint has been followed: for a j-degenerate $\mathbf{i}$-cube $a = e_j c$ we always choose the transversal isomorphism $u = e_j(\varepsilon c)\colon F(e_j(Gc)) \to e_j c$. It follows that each $v_i^\alpha\colon FG(\partial_i^\alpha e_j c) \to F(\partial_i^\alpha e_j Gc)$ is the identity; then $t_i^\alpha\colon G(\partial_i^\alpha e_j c) \to \partial_i^\alpha e_j Gc$ is the identity as well. We (choose to) fill their family with the identity $t\colon e_j Gc \to e_j Gc$, which gives

$$G(e_j c) = e_j Gc, \qquad\qquad \varepsilon(e_j c) = u.Ft = e_j(\varepsilon c).$$

If a is also j'-degenerate, the commutativity of degeneracies ensures that both constructions give the same result. ■

6. Limits and adjoints for cm-categories

We briefly recall the definition of cones and limits from Part II, Section 3, and prove that unitary right adjoints preserve the limits of cm-functors.

6.1. Lift functors. First we recall a tool from II.1.5, II.1.8. For the *positive* integer j there is a *j-directed lift 2-functor* with values in the 2-category of chiral multiple categories indexed by the ordered set $\mathbb{N}|j = \mathbb{N} \setminus \{j\}$, pointed at 0

$$Q_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}. \tag{78}$$

On a cm-category A the cm-category $Q_j\mathsf{A}$ is - loosely speaking - that part of A that contains the index j, reindexed without it:

$$\begin{aligned} &(Q_j\mathsf{A})_{\mathbf{i}} = A_{\mathbf{i}j}, \\ &(\partial_i^\alpha\colon (Q_j\mathsf{A})_{\mathbf{i}} \to (Q_j\mathsf{A})_{\mathbf{i}|i}) = (\partial_i^\alpha\colon A_{\mathbf{i}j} \to A_{\mathbf{i}j|i}), \\ &(e_i\colon (Q_j\mathsf{A})_{\mathbf{i}|i} \to (Q_j\mathsf{A})_{\mathbf{i}}) = (e_i\colon A_{\mathbf{i}j|i} \to A_{\mathbf{i}j}) \qquad (i \in \mathbf{i} \subset \mathbb{N}|j), \end{aligned} \tag{79}$$

and similarly for compositions and comparisons. In the same way for a lax cm-functor $F\colon \mathsf{A} \to \mathsf{B}$ and a transversal transformation $h\colon F \to G\colon \mathsf{A} \to \mathsf{B}$ we let

$$(Q_jF)_{\mathbf{i}} = F_{\mathbf{i}j}, \qquad (Q_jh)_{\mathbf{i}} = h_{\mathbf{i}j} \qquad\qquad (\mathbf{i} \subset \mathbb{N}|j). \tag{80}$$

There is also an obvious restriction 2-functor $R_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}$ where the multiple category $R_j\mathsf{A}$ is that part of A that does not contain the index j. The j-directed faces and degeneracies of A are not used in $Q_j\mathsf{A}$, but yield three natural transformations, also called *faces* and *degeneracy*

$$\begin{aligned} &D_j^\alpha\colon Q_j \to R_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}, \quad (D_j^\alpha)_{\mathbf{i}} = \partial_j^\alpha\colon A_{\mathbf{i}j} \to A_{\mathbf{i}} \qquad (\mathbf{i} \subset \mathbb{N}|j), \\ &E_j\colon R_j \to Q_j\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}, \quad (E_j)_{\mathbf{i}} = e_j\colon A_{\mathbf{i}} \to A_{\mathbf{i}j} \qquad (\mathbf{i} \subset \mathbb{N}|j), \\ &D_j^\alpha E_j = \mathrm{id}. \end{aligned} \tag{81}$$

All the functors Q_j commute. By composing n of them in any order we get an *iterated lift functor* of degree n, in a *positive* direction $\mathbf{i} = \{i_1, ..., i_n\}$

$$Q_{\mathbf{i}} \colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|\mathbf{i}}, \qquad Q_{\mathbf{i}}(\mathsf{A}) = Q_{i_n}...Q_{i_1}(\mathsf{A}),$$
$$\mathrm{tv}_*(Q_{\mathbf{i}}(\mathsf{A})) = \mathrm{tv}_{\mathbf{i}}(\mathsf{A}). \tag{82}$$

6.2. Cones. Let X and A be cm-categories, and let X be small. Consider the *diagonal* functor (of ordinary categories)

$$D \colon \mathrm{tv}_*\mathsf{A} \to \mathrm{Ps}\mathbf{Cmc}(\mathsf{X}, \mathsf{A}). \tag{83}$$

where $\mathrm{tv}_*\mathsf{A}$ is the ordinary category of $\star$-cubes (objects) of A and their transversal maps.

D takes each object A of A to a unitary pseudo functor $\mathsf{X} \to \mathsf{A}$, 'constant' at A via the family of the total $\mathbf{i}$-degeneracies $e_{\mathbf{i}} = e_{i_1}...e_{i_n} \colon A_* \to A_{\mathbf{i}}$

$$\begin{aligned}
&DA \colon \mathsf{X} \to \mathsf{A} \\
&DA(x) = e_{\mathbf{i}}(A), \qquad DA(f) = \mathrm{id}(e_{\mathbf{i}}A) && \text{(for } x \text{ and } f \text{ in } \mathrm{tv}_{\mathbf{i}}\mathsf{X}), \\
&\underline{DA}_i(x) = \mathrm{id}(e_{\mathbf{i}}A) \colon e_i(DA(x)) \to DA(e_i x) && \text{(for } x \text{ in } X_{\mathbf{i}|i}), \\
&\underline{DA}_i(x, y) = \lambda_i \colon e_{\mathbf{i}}(A) +_i e_{\mathbf{i}}(A) \to e_{\mathbf{i}}(A) && \text{(for } i\text{-consecutive cubes } x, y \text{ in } X_{\mathbf{i}}),
\end{aligned} \tag{84}$$

where $\lambda_i = \lambda_i(e_{\mathbf{i}}A) = \rho_i(e_{\mathbf{i}}A)$ is a left and right unitor of A.

Similarly, a $\star$-map $f \colon A \to B$ in A is sent to the constant transversal transformation

$$Df \colon DA \to DB \colon \mathsf{X} \to \mathsf{A}, \qquad (Df)(x) = e_{\mathbf{i}}(f) \colon e_{\mathbf{i}}(A) \to e_{\mathbf{i}}(B) \qquad (x \text{ in } \mathrm{tv}_{\mathbf{i}}\mathsf{X}). \tag{85}$$

Let $T \colon \mathsf{X} \to \mathsf{A}$ be a lax functor. A *(transversal) cone* of T will be a pair $(A, h \colon DA \to T)$ formed of an object A of A (the *vertex* of the cone) and a transversal transformation of lax functors $h \colon DA \to T \colon \mathsf{X} \to \mathsf{A}$; in other words, it is an object of the ordinary comma category $(D{\downarrow}T)$, where T is viewed as an object of the category $\mathrm{Lx}\mathbf{Cmc}(\mathsf{X}, \mathsf{A})$.

By definition (cf. II.1.8), the transversal transformation h amounts to assigning the following data:

- a transversal $\mathbf{i}$-map $hx \colon e_{\mathbf{i}}(A) \to Tx$, for every $\mathbf{i}$-cube x in X,

subject to the following axioms of naturality and coherence:

(tc.1) $\quad Tf.hx = hy \qquad$ (for every $\mathbf{i}$-map $f \colon x \to_0 y$ in X),

(tc.2) h commutes with positive faces, and agrees with positive degeneracies and compositions:

$$\begin{aligned}
&h(\partial_i^\alpha x) = \partial_i^\alpha(hx), && \text{(for } x \text{ in } X_{\mathbf{i}}), \\
&h(e_i x) = \underline{T}_i(x).e_i(hx) \colon e_{\mathbf{i}}(A) \to_0 T(e_i x) && \text{(for } x \text{ in } X_{\mathbf{i}|i}), \\
&h(z) = \underline{T}_i(x, y).(hx +_i hy).\lambda_i^{-1} \colon e_{\mathbf{i}}(A) \to_0 T(z) && \text{(for } z = x +_i y \text{ in } X_{\mathbf{i}}).
\end{aligned}$$

As remarked in II.3.2, a *unitary* lax functor $G \colon \mathsf{A} \to \mathsf{B}$ preserves diagonalisation, in the sense that $G.DA = D(GA)$; therefore G takes a cone $(A, h \colon DA \to T)$ of T to a cone (GA, Gh) of GT.

6.3. Limits of degree zero. As defined in II.3.3, the *(transversal) limit of degree zero* $\lim(T) = (L, t)$ of a lax functor $T\colon \mathsf{X} \to \mathsf{A}$ between chiral multiple categories is a universal cone $(L, t\colon DL \to T)$.

In other words:

(tl.0) L is an object of A and $t\colon DL \to T$ is a transversal transformation of lax functors,

(tl.1) for every cone $(A, h\colon DA \to T)$ there is precisely one $\star$-map $f\colon A \to L$ in A such that $t.Df = h$.

We say that A *has limits of degree zero* on X if all these exist.

Theorem II.3.6 proves that all limits of degree zero in A can be constructed from products, equalisers and *tabulators* - all of degree zero; it also gives a corresponding result for the preservation of such limits by *unitary* lax multiple functors. (Tabulators, the basic form of higher limits, were sketched in Part I and studied in Part II, Section 3.)

6.4. Multiple limits. The general definition of multiple limits in a chiral multiple category A was given in II.4.4.

(a) For a positive multi-index $\mathbf{i} \subset \mathbb{N}$ and a chiral multiple category X we say that A has *limits of type* $\mathbf{i}$ on X if $Q_{\mathbf{i}}\mathsf{A}$ has limits of degree zero on X.

(b) We say that A *has limits of type* $\mathbf{i}$ if this happens for all small chiral multiple categories X.

(c) We say that A *has limits of all degrees* (or *all types*) if this happens for all positive multi-indices $\mathbf{i}$.

(d) We say that A *has multiple limits of all degrees* if all the previous limits exist and are preserved by the multiple functors (see 6.1)

$$D_j^\alpha\colon Q_{\mathbf{i}j}(\mathsf{A}) \to R_jQ_{\mathbf{i}}(\mathsf{A}), \qquad E_j\colon R_jQ_{\mathbf{i}}(\mathsf{A}) \to Q_{\mathbf{i}j}(\mathsf{A}) \qquad\qquad (j \notin \mathbf{i}). \tag{86}$$

In this case, if A is transversally invariant one can always operate a choice of multiple limits such that this preservation is strict.

The Main Theorem of Part II (II.4.5) shows that all multiple limits in A can be constructed from multiple products, multiple equalisers and multiple tabulators; again, it also gives a corresponding result for the preservation of such limits by multiple functors.

We are now able to prove the preservation properties of unitary adjoints.

6.5. Theorem. [Adjoints and limits of degree zero] *Let* $(\eta, \varepsilon)\colon F \dashv G$ *be a colax/lax cm-adjunction, where* both functors are unitary.

Then $G\colon \mathsf{A} \to \mathsf{B}$ *preserves all (the existing) limits of degree zero of lax cm-functors* $T\colon \mathsf{X} \to \mathsf{A}$.

Proof. The argument is the usual one. Let $(A, h\colon D_{\mathsf{A}}(A) \to T)$ be a limit of T in A. We want to prove that the pair $(GA, Gh\colon G.D_{\mathsf{A}}(A) \to GT)$ is a limit of GT in B.

First, since G is unitary, $GD_{\mathsf{A}}(A) = D_{\mathsf{B}}(GA)$ and the pair (GA, Gh) is indeed a cone of the lax cm-functor $GT\colon \mathsf{X} \to \mathsf{B}$.

Moreover, given a cone $(B, k'\colon D_{\mathsf{B}}(a) \to GT)$ of GT, with transversal components $k'x\colon e_{\mathbf{i}}(B) \to GTx$ for every $\mathbf{i}$-cube x in X, the adjunction gives a family $h'x\colon Fe_{\mathbf{i}}(B) \to Tx$, that is a cone $(FB, h'\colon D_{\mathsf{A}}(FB) \to T)$ in A. Therefore there is precisely one transversal map $f\colon FB \to A$ in A such that $h.Df = h'$. This means precisely one transversal map $g\colon B \to GA$ in B such that $Gh.Dg = k'$. ■

6.6. Remark. Since the lift 2-functor $Q_{\mathbf{i}}\colon \mathrm{Lx}\mathbf{Cmc} \to \mathrm{Lx}\mathbf{Cmc}_{\mathbb{N}|j}$ preserves cm-adjunctions, it follows that, if the cm-category A has *multiple limits on* X, these are preserved by a right adjoint cm-functor $G\colon \mathsf{A} \to \mathsf{B}$ (under the previous unitarity assumptions).

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Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146-Genova, Italy

Department of Mathematics and Statistics
Dalhousie University
Halifax NS
Canada B3H 4R2

Email: grandis@dima.unige.it, pare@mathstat.dal.ca