A game theoretic model for improving a railway timetable

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Abstract
The EEC directive 440/91 allows to different transport operators to operate different trains on the same network, so the infrastructure manager has to produce the timetable, taking into account the requests of different agents in a competitive situation.
Subsequently, the transport operators may cooperate, in order to modify their scheduling and increase their income.
The game theoretical approach models this situation in which the agents search for their own utility, rather than for a global maximum, as a coalition formation problem. In particular we refer to the C-Solution proposed by Gerber.
Some examples conclude the paper.
Key words: NTU-game, Coalition formation, Slot allocation.

1 Introduction

The typical framework in the European railways industry in the past century was the presence of monopolistic state-owned companies in all countries. This integrated companies managed both infrastructure and service.

This situation revealed a lot of inefficiencies (deficits, low quality, etc.), so in the 1990s the European Commission decided to open the market to competition. The reform process began with the directive 440/91, followed by different packages of directives [1], in which first of all the Commission asked for the need to separate the management of infrastructure from the management of the service.

In this way the infrastructure become available for several operators through competition in the market or competition for the market. In this view planning the railway timetable and the resolution of possible conflicts between operators become a crucial task.

Precisely, the European Commission emphasizes the concept of non discriminatory access to the railway market. Unfortunately at the current state of art of the research, there does not exist a method to allocate capacity that is able to satisfy this condition.

The literature identifies different methodologies for capacity allocation that we briefly set out afterwards and that draw on auction theory or linear programming.

The increase of competition implies the need to use scientific methods of decision in order to optimize the allocation of available resources, particularly the sequence of trains.

Two other elements that increase the complexity of the problem are the large dimension of the network and the necessity of interconnection with the timetables of boundary countries due to interna-

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[1] We refer to directives EEC 18-19/95, EU 12-13-14/2001 and 49-50-51/2004.

tional trains. For these reasons the problem was approached, at least theoretically, in many different ways, from the classical manual to the most sophisticated software.

The manual approach starts, in general, from the current timetable and tries to improve it, or just to update it, taking into account the requests for insertions and/or eliminations of trains; it should be clear the importance of the previous experiences.

Referring to the sophisticated software approaches it is possible to consider an optimal timetable as the solution of mathematical programming problem whose data consist of the topology of the network (lines and stations), the technical characteristics of the tracks and of the trains, the demand for transport and the utility functions of the trains operators. It is possible to assume as unknown the leaving time and the travel time of each train and consequently to state as constraints the headway times between trains and some requirements arising from the technical characteristics and from the quality of services to the users; under these conditions it is possible to search for the maximum of the global utilities (see *Hooghiemstra et al.*, 1998).

Now, we shortly recall some previous experiences in this field.

In Binary Conflict Ascending Price mechanism (BICAP) (see *Brewer and Plott*, 1996) agents bid to have access to railroad and they are free to increase their bids in order to compete for the available slot. In this procedure there is a central computer that continuously calculates potential allocations. A bid is submitted in real time. The mechanism works like a set of simultaneous ascending auctions. Three elements are involved: a set of feasible outcome allocations, a message space through which agents interact each other and with the allocation authority, and an outcome rule setting how these messages could determine a unique outcome starting from the feasible set of allocations. Each auction is for operating a different train and so there could be as many auctions as there are possible trains.

The allocation process is based on maximizing the total bid from trains considering some feasibility constraints. Agents can increase their bids. In this model each agent could submits bids for trains in a continuous time auction. The mechanism checks if the new bid is higher for a given train that the current bid. The potential allocation of a certain period of time is defined as the set of bids that cause no conflict and maximize the sum of all feasible allocations. Only the highest bids are kept as information by the mechanism. A new allocation is compute and the auction ends when there is no further increase in the bid after a certain period of time.

The experiments carried out by the authors confirm that a decentralized mechanism can solve some of the technical aspects of the rail scheduling problem and yield efficient allocations. Moreover not only the results are efficient, but also the design consistency appears strong.

A different approach was proposed by *Nilsson* (2002) in order to create a train schedule that gets the maximum benefits, addressing two strongly related challenges: The optimisation problem and the incentive problem.

The optimisation problem is concerned with the mathematical aspects inherent in the problem. The incentive problem is referred to the need to make operators reveal their value of track access. The incentive problem is therefore concerned with the need to acquire information about the operators' value-of-access. In order to reach this there is a procedure composed by four steps.

First, the interested operators have to register their preferred trains departures and arrivals. They also include alternatives to the preferred path and submit also a set of bids, one for each alternative path. Each bid indicates the operator's willingness-to-pay (WTP) for the preferred departure-arrival path as well as for the alternative paths.

Then, the IM identifies the value-maximising allocation, i.e. that timetable which generates the largest possible aggregate value of bids. For each train path a set of prices is calculated, given the demand schedules and bids which have been submitted. These are the prices the operators would have to pay in order to run their trains.

In the third step, the information acquired is sent back to operators for further considerations. If there are no conflicts for path between operators, all get their preferred choices and the process can be terminated. If not, one or more operators have not been allocated exactly what they asked for. Step four. These last operators have the opportunity to reconsider their initial train path specifications.

The four step of the process is repeated as long as anyone wants to make changes in bids or departure specifications.

Another important point is that the IM could address problem that are indirectly related to timetabling. The result of the allocation process may also provide other relevant information like that about scarcity of capacity. Using the suggested techniques, scarcity will manifest itself in high track user charges, signalling that users' value of access is high. The process may therefore create relevant information for the investment planning process useful to decide if track supply should be expanded or not.

The paper by *Caprara et al.* (2002) introduces a model based on a graph. They consider a single line and represent each as a departure station and an arrival station; for each of them a set of 1440 nodes, each one corresponding to a different minute of the day, is taken into account. The oriented arcs represent the movement of a train among two stations when connect a node associated to a departure station and a node associated to the following arrival station or the stopping time of a train in a station when connect a node associated to an arrival station and a node associated to the same departure station. A path, i.e. a sequence of arcs from a departure station to an arrival station represents the travel of a train from its origin to its destination. Suitable constraints avoid undesired results. The solution provides a set of paths that maximize the total utility of the trains, on the basis of the difference from their optimal departure time from the originating station (shift, that can be positive if the train leaves later or negative if the train leaves earlier) and of the delay in the travel time (stretch, that is supposed to be only non negative, as the trains ask for the minimal travel time). Further modifications of the basic model allow taking into account multiple tracks in the stations or simple networks, like a line with two branches *Caprara et al.* (2006).

Recently, *Borndörfer et al.* (2005) used an iterative combinatorial auction that takes place in a sequel of round. Each round consists of two stages. In the first stage, each operator submits simultaneously a set of bids that, jointly with the set of standing bids, represents the set of "live bids". In the second stage the "optimization machinery" is applied on the set of live bids and computes the set of bids accepted in the round, that become the new set of standing bids. During the optimisation process the auctioneer must determine a conflict-free slot schedule that maximizes the network proceeds. They denote optimal allocation problem (OPTRA) the winner determination problem that is a so called multi-commodity flow problem with additional constraints solved with integer programming techniques.

The action ends if for a fixed number of rounds, the total proceed does not change.

We assume that the TOs communicate a small set of data consisting of the ideal departure time and a time window that represents the time interval in which it is still profitable to schedule the departure of the train. At this point the IM uses for all the trains a standard continuous piecewise linear utility function that assigns a fixed maximum value to the ideal departure time and null utility outside of the time window (see the examples in Section 3): in this way it is possible to have a complete set of (approximate) data and solve the problem of the scheduling.

TOs may decide to cooperate in order to improve their utilities so they set available more details about their utility functions. Using better information it is possible that they modify their scheduling (without influencing the timetable of the other trains), increasing their profits.

We may think of a utility function as a map that assigns to every possible leaving instant a real number that represents the income for operating the train according to this departure time.

Of course it is improbable to have exact data, but it is possible to have a good approximation referring to the situation of the previous years and taking into account the opinions of the users; on the other hand the utility function of a train can be affected if the final timetable does not respect the forecast about other trains. Let us illustrate it using a simple example: suppose that the train A considers its best choice to leave from a given station ten minutes after the arrival of the train B; in this situation the maximum of the utility is defined according to the arrival time of train B in the previous timetable, that can be no longer valid in the new one.

In this paper we propose a game theoretical approach in order to model the previous competitive situation in which the agents are interested in improving their own utility, rather than the global one; when the TOs decide to cooperate we have a problem of coalition formation. This problem often arises when we want to model a real-world situation; in fact it is very important to design a model taking into account the various possibilities of the players, in particular that it is not necessary that all the players form the grand coalition and that each player decides to cooperate with other players not for the worth of the coalition but on the basis of its own payoff (see *Hart and Kurz*, 1983 and *Greenberg*, 1994). We are interested in getting also a numerical solution of the problem, so we refer to the C-Solution proposed by *Gerber* (2000), whose main characteristic is that it does not necessarily favour the formation of the grand coalition; the third section is devolved to this aim.

In Section 4 we present some examples. We start with an example in which the players can transfer their utility; then we drop this assumption and give three more examples: the first one considers different classical solutions for bargaining problem; in the second one the solution proposed by the IM results to be optimal (in the sense of Pareto); the last example introduces new questions that arise when the bargaining region is not convex.

Finally, Section 5 concludes.

2 Definitions

In this section we introduce some definitions that will be used through the paper. We start by defining a cooperative game without transferable utility (NTU-game).

Definition 1 *A cooperative game without transferable utility is a couple $G = (N, V)$, where $N = \{1, ..., n\}$ is the player set and V is the characteristic function that maps each coalition S in the set of its feasible payoffs, s.t.:*

- $V(S) \subset \mathbb{R}^S$;
- *$V(S)$ is closed and non-empty;*
- *$V(S) = V(S) - \mathbb{R}^S_{\geq}$ (comprehensiveness).*

Another important element in this paper is the bargaining problem, introduced by *Nash* (1950) with two players and generalized as follows.

Definition 2 *A bargaining problem is a couple (F, d), where F is the feasibility set, i.e. the closed, convex, bounded and non-empty set of payoffs which players can agreed on after bargaining and $d = (d_1, ..., d_n), d \in \mathbb{R}^n$ is the disagreement point, i.e. the starting payoff or the minimum that players can guarantee if they do not reach an agreement.*

Many solutions were proposed for a bargaining problem; here we recall three classical ones:

- the Nash solution (1950):

$$\mathrm{N}(F, d) = argmax \left\{ \prod_{i \in N} (x_i - d_i) \mid x \in F,\ x \geq d \right\}$$

- the Kalai-Smorodinsky solution (1975):

$$\mathrm{KS}(F,d) = argmax\left\{\frac{x_1 - d_1}{a_1 - d_1} = ... = \frac{x_n - d_n}{a_n - d_n} \mid x \in F,\ x \geq d\right\}$$

  where $a_i = max\{x_i \in \mathbb{R} \mid x \in F,\ x \geq d\}, i \in N$

- the Egalitarian solution (cf. *Roth*, 1979):

$$\mathrm{Eg}(F,d) = argmax\{x_1 - d_1 = ... = x_n - d_n \mid x \in F,\ x \geq d\}$$

3 C-Solution

In this section we present the coalition formation method of *Gerber* (2000) that refers to the dynamic solution of a suitable abstract game (*Shenoy*, 1980). A nice characteristic of this process is that the final numerical solution, the C-Solution, specifies not only which coalitions form, but assign a payoff to each player, starting from its "power" in the other coalitions.

3.1 Abstract Games

We recall some basic concepts about abstract games, which are the simplest idea of a game.

- An *abstract game* is a couple (X, dom) where X is an arbitrary set and $dom \subset X \times X$ is a binary relation on X, called domination; given $x, y \in X$, x is accessible from y, or $y \to x$, if there exist $z_1, \ldots, z_m \in X$ s.t.

$$x = z_1\ dom\ z_2\ dom\ \ldots\ dom\ z_{m-1}\ dom\ z_m = y$$

- An *elementary dynamic solution* is a set $S \subseteq X$ that satisfies the following conditions:
  1. $x \in S, y \in X \setminus S \Rightarrow x \not\to y$ (elements not in S are not accessible from elements of S).
  2. $x, y \in S \Rightarrow x \leftrightarrow y$ (two elements of S are accessible one another).

- A *dynamic solution* is a set P that is union of elementary dynamic solutions.

- For dynamic solutions the following theorem holds: If X is finite, then P is non-empty.

3.2 Assumptions

In the following we suppose (cf. *Gerber*, 1996) that:

- An NTU-game is a "menu" of pure bargaining games $H^S = \{(F^S, d^S)\}$ for every coalition S, where F^S is the feasible region and d^S is the disagreement point.

- Each player can join to one coalition.

- The players bargain over utility distribution in each game.

In practice the players have to face two intertwined decision problems: which coalition to form and which payoff vector to choose from the bargaining region for the members of each coalition. Other approaches to coalition formation problems mainly search for an allocation of the worth of the grand coalition (see *Bennett and Zame*, 1988).

3.3 Tools

- A *bargaining function* is a function $\phi^S : H^S \to \mathbb{R}^{\mathbb{N}}_{\mathbb{S}}$ that assigns a payoff vector as a solution for each bargaining problem for S, and satisfies the following properties:

$$\begin{array}{ll} \textit{Feasibility}: & \phi^S(F^S, d^S) \in F^S \\ \textit{Individual rationality}: & \phi^S(F^S, d^S) \geq d^S \\ \textit{Pareto Optimality}: & x \in \mathbb{R}^{\mathbb{N}}_{\mathbb{S}}, \curvearrowleft > \phi^{\mathbb{S}}(\mathbb{F}^{\mathbb{S}}, {}^{\mathbb{S}}) \Rightarrow \curvearrowleft \notin \mathbb{F}^{\mathbb{S}} \end{array}$$

  The solutions presented in Section 2 are possible choices.

- A *payoff configuration* is a couple $(P, x) \in \bigcup_{P \in \Pi}(\{P\} \times F_V(P))$ where $P \in \Pi$ is a coalition structure, i.e. a partition of the player set N, and $F_V(P)$ is the set: $\{x \in \mathbb{R}^{\mathbb{N}} \mid \curvearrowleft \in \mathbb{V}(\mathbb{S}), \forall \mathbb{S} \in \mathbb{P}\}$.

- The *domination* relation between two different payoff configurations $(P^1, x^1), (P^2, x^2)$ is:

$$(P^1, x^1)\ dom\ (P^2, x^2) \Leftrightarrow \exists\ R \in P^1 \mid x^1_i > x^2_i, \forall\ i \in R$$

- The set of *decisive coalitions* for a game V, or $\mathcal{E}^V$, is the set of coalitions $S, |S| \geq 2$, s.t.:

$$\exists\ y \in V(S) \mid y > \underline{x}_S$$

  where $x_S = (\underline{x}_i)_{i \in S}$ and $\underline{x}_i = sup\ \{t \in \mathbb{R} \mid \approx^{\sqsupset} \in \mathbb{V}(\sqsupset)\}$

- A *reduced game* w.r.t. the coalition S, or V^{-S}, is the game:

$$V^{-S}(T) = \begin{cases} V(T) & \text{if } T \neq S \\ \{y \in \mathbb{R}^{\mathbb{N}}_{\mathbb{T}} \mid \curvearrowright \leq \underline{\frown}_{\mathbb{T}}\} & \text{if } T = S \end{cases}$$

- A *feasibility function* is a function $d^S_V : \{x \in \mathbb{R}^{\mathbb{N}}_{\mathbb{S}} \mid \curvearrowleft \geq \underline{\frown}_{\mathbb{S}}\} \to \mathbb{V}(\mathbb{S})$ that returns a feasible disagreement point if the given one is infeasible; it has to satisfy the following properties:

  1. $d^S_V(x) \geq \underline{x}_S$
  2. $d^S_V(x) = x$, if $x \in V(S)$

  A possible choice is:

$$(d^S_V(x))_i = \begin{cases} 0 & \text{if } i \notin S \\ x_i & \text{if } i \in S \text{ and } x \in V(S) \\ max\ \{\underline{x}_i, max\ \{t \in \mathbb{R} \mid (\approx, \curvearrowleft_{\sqsupset}) \in \mathbb{V}(\mathbb{S})\}\} & \text{if } i \in S \text{ and } x \notin V(S) \end{cases}$$

3.4 Computation of the C-Solution

The C-Solution concept is defined by induction on the cardinality of the set of decisive coalitions $\mathcal{E}^V$; in other words we consider first the coalition structure in which each player stands alone, next we define an abstract game where X is the set of payoff configurations corresponding to the coalition structures generated by adding at each step one of the decisive coalitions and considering as payoff vector the solution of the bargaining problem for the decisive coalitions and the stand-alone solution for the singletons and using the above defined domination relation.
Formally we have:

Initial step $\quad |\mathcal{E}^V| = 0 \ \Rightarrow \mathcal{E}^V = \emptyset$
The C-Solution is $(\{\{1\}, \{2\}, \dots, \{n\}\}, (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n))$

Iterative step $\quad |\mathcal{E}^V| = m \geq 1$
The C-Solution is the dynamic solution of the abstract game (X, dom) where:

$$X = \left\{ (P,x) \in \Pi \times \mathbb{R}^{\mathbb{N}} \mid \mathbb{T} \in \mathbb{P} \Rightarrow \left\{ \begin{array}{l} T \in \mathcal{E}^V \text{ and } x_T = \phi^T(V(T), d_V^T(y^T))\big|_T \\ \text{or} \\ T = \{i\} \text{ and } x_i = \underline{x}_i \end{array} \right. \right\}$$

and *dom* is the domination defined above.

The point y^T is the average payoff of the players in the C-Solution $\{(P^1, x^1), \ldots, (P^{K(T)}, x^{K(T)})\}$ of the reduced game V^{-T}, i.e. the opportunities of the players in the other coalitions:

$$y_j^T = \left\{ \begin{array}{ll} \frac{1}{K(T)} \sum_{h=1}^{K(T)} x_j^h & \text{if } j \in T \\ 0 & \text{if } j \notin T \end{array} \right.$$

Note that V^{-T} is originated according to the current $\mathcal{E}^V$, so by the inductive hypothesis the C-Solution was computed in the previous step; as X is finite, the dynamic solution is always non-empty.

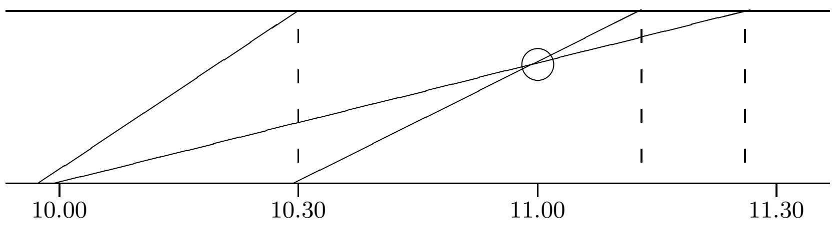
4 Examples

In this section we present four examples. Example 1 refers to a situation with three TOs that have required conflicting departure times. The solution proposed by the IM may be improved when the TOs take into account their utility functions. The environment of this example allows for transferable utilities among the agents, so that they look for the maximal utility and the proposed methodology allows computing also the monetary compensations that the agents have to pay or receive as a consequence of the agreement. The appendix at the end of the paper contains all the steps of the iterative procedure for the case in Example 1. Examples 2, 3 and 4 consider a situation in which the agents are not allowed for side payments. In these cases the improvement of the solution is obtained through a bargaining procedure among the agents. In order to have a simple graphical representation of the strategies and of the utilities of the agents we restrict to two-agent situations. The starting point, i.e. requirements of the agents are the same in all three examples, but introducing different real utility functions we analyse three possible situations: in Example 2 the agents may improve their utilities w.r.t. the proposal of the IM and we show three possible alternatives corresponding to the bargaining problem solutions of Nash, Kalai-Smorodinsky and Egalitarian; in Example 3 the agents realize that the proposal of the IM is already Pareto optimal, i.e. cannot be improved by an agent without a reduction of the utility of the other one; finally, Example 4 shows that under suitable utility functions the agents have to face a situation in which the bargaining region is non-convex, in contrast with the standard literature. The situation of Example 4 allows us to propose some non-standard approaches to this kind of problems. In all the examples we assume that the headway time between trains is at least two minutes and that the travel time is fixed, so the problem requires only determining the departure time.

Example 1 - Trains with transferable utilities
In this example we suppose that in the computation of the C-Solution, in each bargaining problem H^S, $S \subseteq N$ the players in S agree on the scheduling that corresponds to their maximal global utility $M(S)$, so $V(S) = \{x \mid \sum_{i \in N} x_i \le M(S)\}$. Under this hypothesis the Nash, Kalai-Smorodinsky and Egalitarian solutions coincide; the corresponding utilities are obtained with side payments.

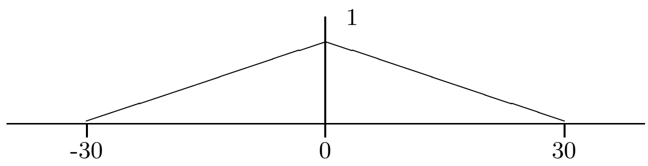
At first we have the requirement of TOs:

$$\begin{array}{llll} p_1 = \ \ 9.58 & t_1 = 32 & a_1 = 10.30 & f_1 = [-30, 30] \\ p_2 = 10.00 & t_2 = 86 & a_2 = 11.26 & f_2 = [-30, 30] \\ p_3 = 10.30 & t_3 = 43 & a_3 = 11.13 & f_3 = [-30, 30] \end{array}$$



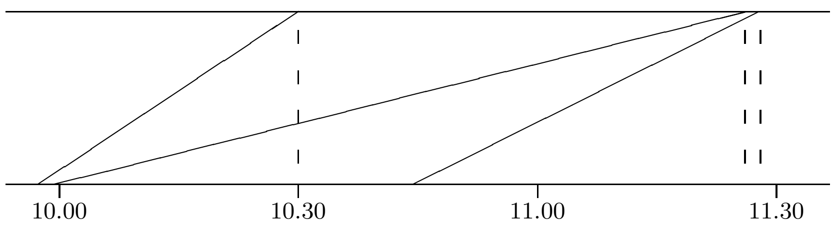
In order to solve the conflict between the players 2 and 3, the IM assigns to the TOs the following standard utility function:

$$\hat{u}(x_i) = \begin{cases} 1 - \frac{|x_i|}{30} & \textit{if } |x_i| \leq 30 \\ 0 & \textit{otherwise} \end{cases}$$



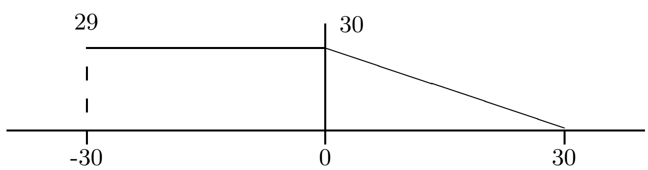
Referring to these utility functions we suppose that the IM searches for the maximum of the total utility $\left(\sum_{i=1,\ldots,3} \hat{u}_i(x)\right)$, *minimizing the losses of the trains. The optimal solution corresponds to the following departure times:*

$$\begin{aligned} p_1 &= \ \ 9.58 \\ p_2 &= 10.00 \\ p_3 &= 10.45 \end{aligned}$$



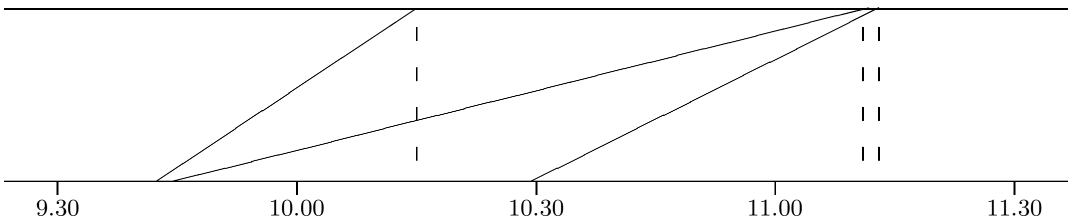
Now we take into account the real utility functions of TOs:

$$u_i(x_i) = \begin{cases} 30 + \frac{x_i}{30} & \textit{if } -30 \leq x_i \leq 0 \\ 30 - x_i & \textit{if } \quad 0 < x_i \leq 30 \end{cases}$$



The real utilities of the three players are 30,30,15 respectively. This situation is a natural habitat for our purpose: it is possible that the real utility functions induce a better solution for the three players. In fact computing the C-Solution (see Appendix) we obtain the following scheduling:

$$\begin{aligned} p_1 &= \ \ 9.43 \\ p_2 &= \ \ 9.45 \\ p_3 &= 10.30 \end{aligned}$$



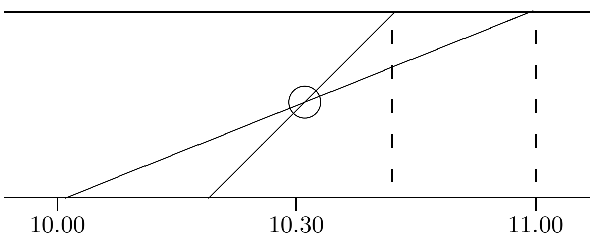
The meaning of the solution is the following: Train 1 leaves 15 minutes early with a utility of 29.50 but receive 1.16 from player 3 so its final payoff is 30.66; Train 2 leaves 15 minutes early with a utility of 29.50 but receive 7.16 from player 3 so its final payoff is 36.66; Train 3 leaves at its ideal departure time with a utility of 30.00 and pays 8.83 to players 1 and 2 so its final payoff is 21.66. Note that all the players have increased their payoffs. □

Example 2 - Trains without transferable utilities
In order to have a better analysis of the bargaining problem we consider that players cannot transfer their utilities. Starting from the strategies of each player, i.e. the possible departure times, we build up the space of strategies. If a situation of conflict shows up it means that some tuples of strategies are infeasible.

We define the bargaining problem using the utility functions of the players in order to define the bargaining region in the space of the utilities; in this case with two trains the disagreement point is the solution proposed by the IM. We analyze the Nash, Kalai-Smorodinsky the Egalitarian solutions.

Consider the following requirements of TOs:

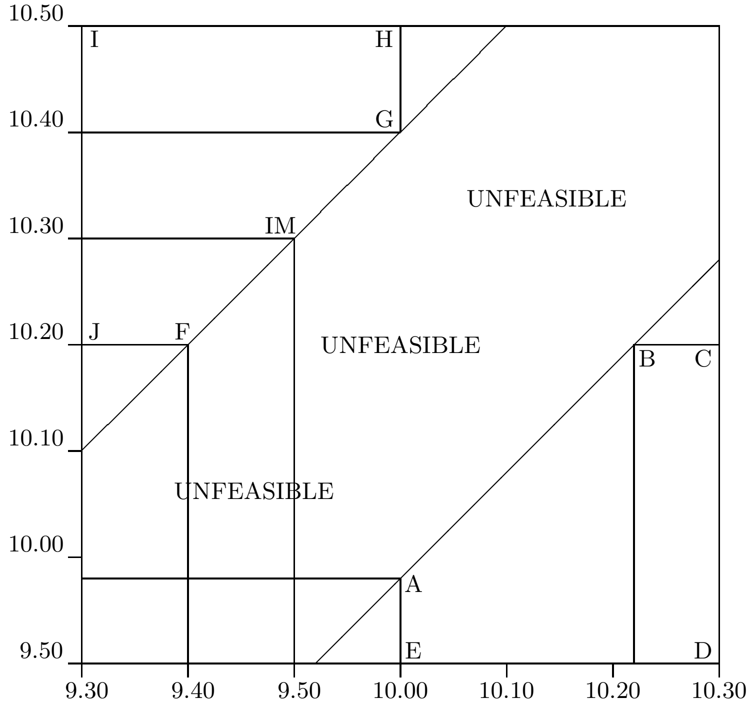
$$
\begin{array}{llll}
p_1 = 10.00 & t_1 = 60 & a_1 = 11.00 & f_1 = [-30, 30] \\
p_2 = 10.20 & t_2 = 22 & a_2 = 10.42 & f_2 = [-30, 30]
\end{array}
$$



Using the standard utility functions, maximizing the total utility and minimizing the loss of each train the IM solves the conflict between the trains with the following scheduling:

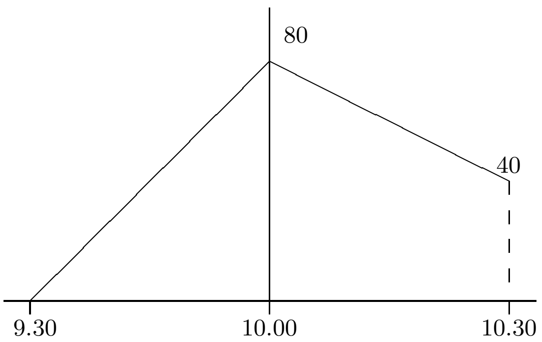
$$
\begin{array}{l}
p_1 = \ \ 9.50 \\
p_2 = 10.30
\end{array}
$$

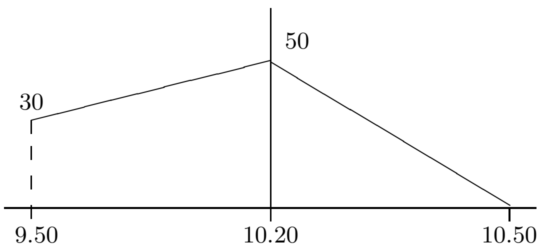
The space of strategies of the two players for the problem is shown below:



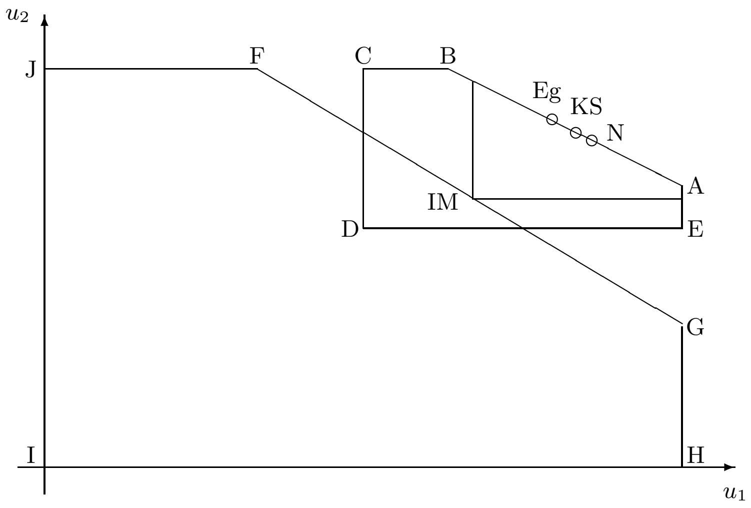
where the points A, B, C, D, E, F, G, H, I, J *represent the extreme points that will be used to define the bargaining region in the space of utilities of the two players and the point* IM *represents the solution proposed by the IM.*

Introducing the real utility functions of the two TOs:





we can build up the related bargaining problem:



where the utilities of the two players in the different points are:

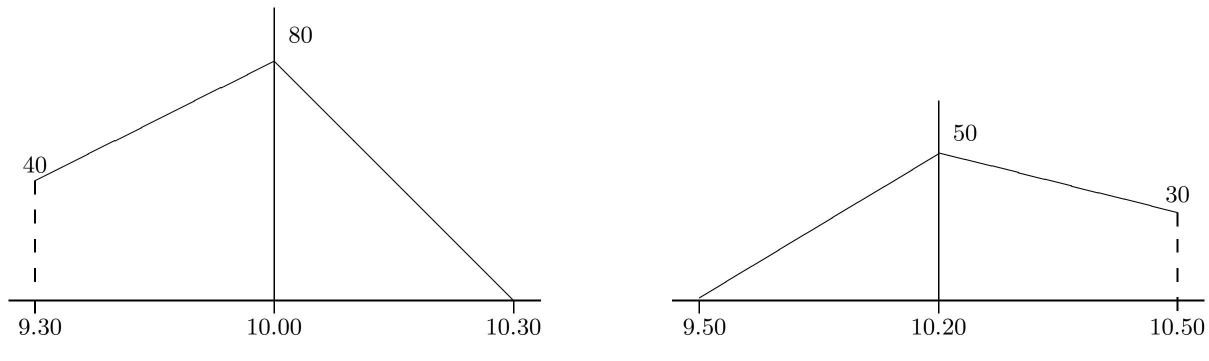
	A	B	C	D	E	F	G	H	I	J	IM	N	KS	Eg
u_1	80	$50\frac{2}{3}$	40	40	80	$26\frac{2}{3}$	80	80	0	0	$53\frac{1}{3}$	$68\frac{2}{3}$	$66\frac{2}{3}$	$63\frac{14}{25}$
u_2	$35\frac{1}{3}$	50	50	30	30	50	$16\frac{2}{3}$	0	0	50	$33\frac{1}{3}$	41	42	$43\frac{14}{25}$

The timetables corresponding to the three solutions (where seconds are in brackets) are:

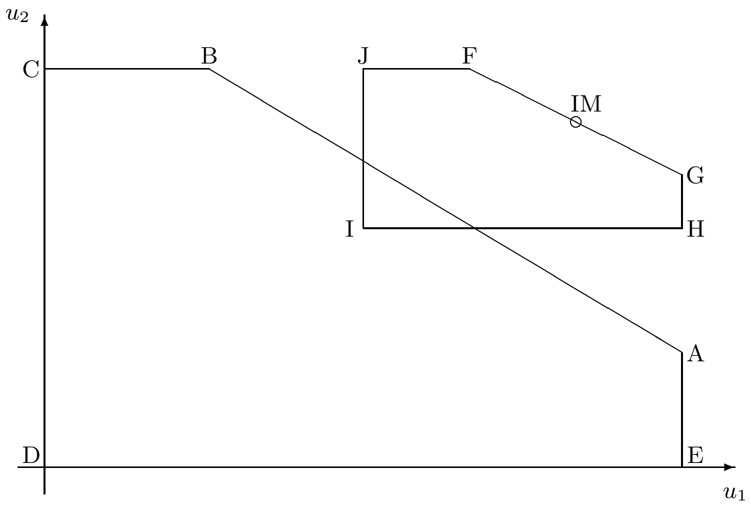
$$
\begin{array}{lll}
\text{N} & p_1 = 10.08(.30) & \approx 10.08 \\
 & p_2 = 10.06(.30) & \approx 10.06 \\
\text{KS} & p_1 = 10.10.00 & = 10.10 \\
 & p_2 = 10.08.00 & = 10.08 \\
\text{Eg} & p_1 = 10.12(.20) & \approx 10.12 \\
 & P_2 = 10.10(.20) & \approx 10.10
\end{array}
$$

□

Example 3 - Pareto optimal disagreement point
Consider now the following real utility functions of the two TOs:



and the new bargaining problem:

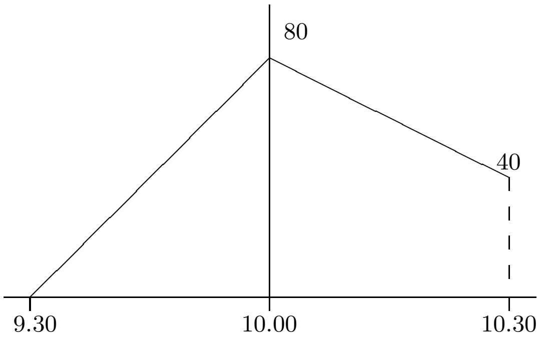


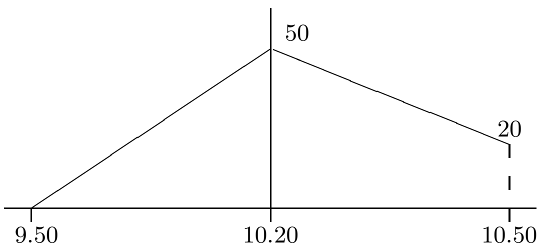
where the utilities of the two players in the different points are:

	A	B	C	D	E	F	G	H	I	J	IM
u_1	80	$21\frac{1}{3}$	0	0	80	$53\frac{1}{3}$	80	80	40	40	$66\frac{2}{3}$
u_2	$13\frac{1}{3}$	50	50	0	0	50	$36\frac{2}{3}$	30	30	50	$43\frac{1}{3}$

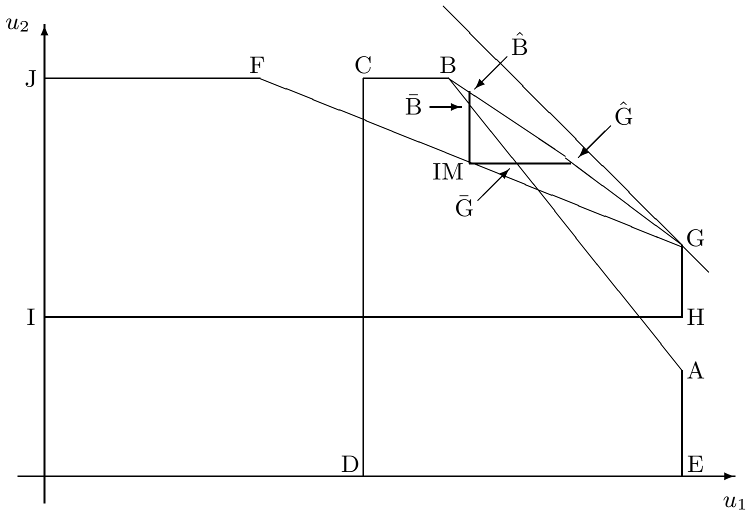
In this case the solution proposed by the IM is already on the Pareto boundary of the bargaining region and is optimal (we can improve it for one player only with a loss for the other one). □

Example 4 - Non-convex bargaining region
A very interesting problem appears when we consider a slightly different situation in which the real utility functions of the two TOs are:





The bargaining problem is the following:



where the utilities of the two players in the different points are:

	A	B	C	D	E	F	G	H	I	J	IM	$\hat{\mathrm{B}}$	$\hat{\mathrm{G}}$	$\bar{\mathrm{B}}$	$\bar{\mathrm{G}}$
u_1	80	$50\frac{2}{3}$	40	40	80	$26\frac{2}{3}$	80	80	0	0	$53\frac{1}{3}$	$53\frac{1}{3}$	$65\frac{1}{3}$	$53\frac{1}{3}$	$58\frac{2}{3}$
u_2	$13\frac{1}{3}$	50	50	0	0	50	30	20	20	50	40	$48\frac{2}{11}$	40	$46\frac{2}{3}$	40

Here the bargaining region is not convex. If we consider also the area delimited by the segment BG *as the usual theory of bargaining suggests a new question arises: what is the meaning of these new points, or more precisely the points of the individually rational Pareto boundary* $\hat{\mathrm{B}}\hat{\mathrm{G}}$*?*
In the space of strategies the segment $\hat{\mathrm{B}}\hat{\mathrm{G}}$ *corresponds to infeasible strategies. We can consider these points as corresponding to correlated strategies or to lotteries of the two (feasible) events* B *and* G*. A correlated strategy is impossible in the present situation because the timetable is fixed for all the season and cannot be varied on the basis of a probabilistic event; on the other hand a lottery requires that the players accept the risk of a result that can be worse of the solution of the IM. Another possible interpretation corresponds to a side payment between the players, but in this case the two players will try to maximize their joint utility so they choose* G *and if they agree on side payments we are back to the situation of Example 1. Finally we can restrict to consider only the individually rational Pareto boundary* $\bar{\mathrm{B}}\bar{\mathrm{G}}$*, returning to a usual bargaining problem, as in Example 2.* □

5 Concluding Remarks

The approach proposed in this paper allows improvements of the railway timetable, when a small set of agents decide to cooperate exchanging a larger set of data w.r.t. the dataset communicated to the IM. The C-solution method allows determining not only a better scheduling, but also how to compensate the agents that accept a worst scheduling, in order to increase the global utility of the cooperating agents. Clearly under this assumption it is possible to have the maximal utility, with positive influence on the quality of service for the users. Possible further developments of the present research are towards the analysis of strategic behaviour of the agents in a combinatorial auction, following the ideas first addressed by *Rassenti et al.*(1982) and reconsidered by *Borndörfer et al.* (2005). Another point deserving better analysis is the situation shown in the last example. In fact the assumption of convexity for the bargaining region is important from a theoretical point of view, but it may result no longer valid in this and other real-world situations.

Appendix

Computation of the C-Solution for Example 1
The characteristic function V for each coalition is defined referring to the solution proposed by the IM and to the real utility functions of the players:

$$V(1) = \{x \mid x_1 \leq 30\};\ V(2) = \{x \mid x_2 \leq 30\};\ V(3) = \{x \mid x_3 \leq 15\}$$
$$V(12) = \{x \mid x_1 + x_2 \leq 60\};\ V(13) = \{x \mid x_1 + x_3 \leq 45\};\ V(23) = \{x \mid x_2 + x_3 \leq 57\}$$
$$V(123) = \{x \mid x_1 + x_2 + x_3 \leq 89\}$$

The set of decisive coalitions is:

$$\mathcal{E}^V = \{\{23\}, \{123\}\}$$

Initial step
$|\mathcal{E}^V| = 0$ $\quad \mathcal{E}^V = \emptyset$
The unique coalition structure is {1, 2, 3}.
The C-Solution is ({1, 2, 3}, (30, 30, 15)), according to the scheduling proposed by the IM.

Iterative step
$|\mathcal{E}^V| = 1$ $\quad \mathcal{E}^V = \{\{23\}\}$
There are two coalition structures: {1, 2, 3} and {1, 23}.
The payoff vector is computed w.r.t. the coalition {23}:
T = {23} $\quad y^T = (0, 30, 15)$ is feasible
$\phi^T(V(T), (y^T)) = (0, 36, 21)$
X = {({1, 2, 3}, (30, 30, 15)), ({1, 23}, (30, 36, 21))}
The C-Solution is ({1, 23}, (30, 36, 21)), i.e. train 1 leaves at 9.58 (IM solution), train 2 at 10.02 and 3 at 10.00.

$\mathcal{E}^V = \{\{123\}\}$
There are two coalition structures: {1, 2, 3} and {123}.
The payoff vector is computed w.r.t. the coalition {123}:
T = {123} $\quad y^T = (30, 30, 15)$ is feasible
$\phi^T(V(T), (y^T)) = (34.66, 34.66, 19.66)$
X = {({1, 2, 3}, (30, 30, 15)), ({123}, (34.66, 34.66, 19.66))}
The C-Solution is ({123}, (34.66, 34.66, 19.66)), i.e. train 1 leaves at 9.43, train 2 at 9.45 and 3 at 10.30.

$|\mathcal{E}^V| = 2$ $\quad \mathcal{E}^V = \{\{23\}, \{123\}\}$
There are three coalition structures: {1, 2, 3}, {1, 23} and {123}.
The payoff vectors computed w.r.t. the coalitions {23} and {123} are:
T = {23} $\quad y^T = (0, 34.66, 19.66)$ is feasible
$\phi^T(V(T), (y^T)) = (0, 36, 21)$
T = {123} $\quad y^T = (30, 36, 21)$ is feasible
$\phi^T(V(T), (y^T)) = (30.66, 36.66, 21.66)$
X = {({1, 2, 3}, (30, 30, 15)), ({1, 23}, (30, 36, 21)),
({123}, (30.66, 36.66, 21.66))}
The C-Solution is ({123}, (30.66, 36.66, 21.66)), i.e. train 1 leaves at 9.43, train 2 at 9.45 and 3 at 10.30, but there is a different transfer of utilities among the players.

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