COHERENCE AND DISTRIBUTIVE LATTICES IN HOMOLOGICAL ALGEBRA

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Abstract. Complex systems in homological algebra present problems of coherence that can be solved by proving the distributivity of the sublattices of subobjects generated by the system. The main applications deal with spectral sequences, but the goal of this paper is to convey the importance of distributive lattices (of subobjects) in homological algebra, to researchers outside of this field; a parallel role played by orthodox semigroups (of endorelations) is referred to but not developed here.

Introduction

This brief article is about coherence in homological algebra, for abelian groups (or more generally for abelian categories). The main applications deal with spectral sequences, but the core of the article only needs the elementary theory of abelian groups and lattices.

Various parts of homological algebra, from the homology of chain complexes to the theory of spectral sequences, are based on 'induction on subquotients' (i.e. quotients of subobjects, or - equivalently - subobjects of quotients). However, the coherence of this procedure of induction makes serious problems, that are often overlooked. The reader may recall that Mac Lane's book on 'Homology' [M1] gives some sufficient conditions that ensure that induction is consistent with composition, in Proposition II.6.3; but these conditions are *not* invariant under inverting the induced isomorphisms.

Problems may already arise in the simplest situation, *canonical isomorphisms* between subquotients of the same object (induced by the identity of the latter): first, such isomorphisms need not be closed under composition; second, if we extend them in this sense the result need not be uniquely determined (as we show in Section 3). Yet, such isomorphisms are frequently used when working with complicated systems, in particular those that give rise to spectral sequences.

Indeed, the solution of the coherence problem depends on the distributivity of the lattices of subobjects generated by the system that we are studying. We prove here, in Section 6, the following theorem.

Given a sublattice X of the (modular) lattice of all subgroups of an abelian group A, let us consider the subquotients of A with numerator and denominator belonging to X. Then the canonical isomorphisms among these subquotients are closed under composition (and form a coherent system) if and only if the lattice X is distributive.

Work supported by a contract of Università di Genova.

2000 Mathematics Subject Classification: 118G99, 06A35,18G40, 55HXX, 20M19.

Key words and phrases: homological algebra, distributive lattice, spectral sequence, orthodox semigroup.

This is a reduced, elementary form of our 'Coherence theorem for homological algebra'. A more complete form of the theorem will be mentioned, without proof, at the end of the article, in Section 10. It was proved by the author in a series of papers of the 1970's ([G2] - [G8]), and used in various papers on 'Distributive homological algebra', like [G9] - [G12]. A self-contained proof will be given in a book in preparation [G13].

The papers mentioned above prove that the following systems are 'distributive' (loosely speaking, this means that they always generate distributive lattices of subobjects), and *their coherence is automatically satisfied*:

- bifiltered object,

- sequence of morphisms,

- filtered chain complex,

- double complex,

- Massey's exact couple [Ma],

- Eilenberg's exact system [Ei].

The same property of distributivity also permits representations of these structures by means of sets and *lattices of subsets*; this yields a precise foundation for the heuristic tool of Zeeman diagrams [Ze, HiW], as universal models of spectral sequences (see [G9] - [G13]).

On the other hand, the following systems are *not* distributive:

- trifiltered object,

- bifiltered chain complex,

and we show in Section 8 a strong form of inconsistency in the latter, that can lead to gross errors if the interaction of the two spectral sequences is explored further than it is normally done.

Note that the two spectral sequences of a double complex are usually defined by means of the associated total complex and its two filtrations. Yet, as claimed above, the double complex *is* a distributive system where a *consistent theory of two interacting spectral sequences exists* - provided such sequences are defined directly, without introducing direct sums and the total complex (Section 9).

This article develops part of a conference at CatAlg2011, Gargnano (Italy), in September 2011.

Literature. Spectral sequences in abelian categories and their applications have been treated in many works on homological algebra and algebraic topology, like [Ei, CE, Gt, Hu, HiW, M1, Sp, Am, HiS, Sw, We, Mc]. Zeeman diagrams for spectral sequences, introduced in [Ze], are also expounded in the book by Hilton and Wylie [HiW]. Birkhoff's and Grätzer's books [Bi, Gr] are standard references for lattice theory.

As remarked in Section 10, the coherence theorem draws new, tight links between homological algebra, lattice theory and the theory of regular, orthodox, quasi-inverse and inverse semigroups. References for the last topics are given in Section 10.

Terminology. Lattice always means here an ordered set with finite joins and meets, which *include* the least and greatest element: the empty join and meet, respectively. A sublattice of a lattice has the same least and greatest element; we use the term 'quasi sublattice' when we only want to express closure under binary joins and meets. The symbol $\subset$ always denotes *weak* inclusion (of subsets, subgroups, etc.).

1. Subquotients and regular induction

For the sake of simplicity, we work in the category Ab of abelian groups, but everything can be extended to abelian categories and even further (see Section 10).

A subquotient $S = M/N$ of an abelian group A is a quotient of a subobject (M) of A, or equivalently a subobject of a quotient (A/N). It is determined by two subgroups $N \subset M$ of A, via a commutative square that is bicartesian, i.e. pullback and pushout:

$$\begin{tikzcd}
M \arrow[r, tail, "m"] \arrow[d, two heads, "p"'] & A \arrow[d, two heads, "q"] \\
S \arrow[r, tail, "n"'] & A/N
\end{tikzcd} \tag{1}$$

The prime example, of course, is the homology subquotient $H = \mathrm{Ker}\,\partial/\mathrm{Im}\,\partial$ of a differential group (A, ∂).

A homomorphism $f \colon A \to B$ is given. If M and H are subobjects of A and B respectively, and $f(M) \subset H$, we have a commutative diagram with short exact rows

$$\begin{tikzcd}
M \arrow[r, tail, "m"] \arrow[d, "f'"'] & A \arrow[r, two heads, "p"] \arrow[d, "f"'] & A/M \arrow[d, "f''"] \\
H \arrow[r, tail, "h"'] & B \arrow[r, two heads, "u"'] & B/H
\end{tikzcd} \tag{2}$$

More generally, given two subquotients M/N of A and H/K of B, we say that f *has a regular induction* from M/N to H/K whenever

$$f(M) \subset H \quad \text{and} \quad f(N) \subset K. \tag{3}$$

Then, we have a *regularly induced* homomorphism $g \colon M/N \to H/K$. In fact, one can form the left diagram below, by applying (2) in two different ways

$$\begin{tikzcd}
M \arrow[r, tail, "m"] \arrow[d, "f'"'] & A \arrow[r, two heads, "q"] \arrow[d, "f"'] & A/N \arrow[d, "f''"] \\
H \arrow[r, tail, "h"'] & B \arrow[r, two heads, "v"'] & B/K
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, two heads, "q'"] \arrow[d, "f'"'] & M/N \arrow[r, tail, "m'"] \arrow[d, "f"'] & A/N \arrow[d, "f''"] \\
H \arrow[r, two heads, "v'"'] & H/K \arrow[r, tail, "h'"'] & B/K
\end{tikzcd} \tag{4}$$

Then we get the right diagram above (and the homomorphism g), by epi-mono factorisation of the rows of the left diagram.

Regular induction is (obviously) preserved by composition. But it is a too restricted notion.

2. Canonical isomorphisms

We now use the category RelAb of (additive) *relations*, or (additive) *correspondences*, between abelian groups (cf. Mac Lane [M1], Hilton [Hi]).

A relation $a\colon A \to B$ is a subgroup of the direct sum $A \oplus B$. It should be viewed as a 'partially defined, multi-valued homomorphism', that sends an element $x \in A$ to the subset $\{y \in B \mid (x, y) \in a\}$ of B. The composite of a with $b\colon B \to C$ is

$$ba = \{(x, z) \in A \oplus C \mid \exists y \in B :\ (x, y) \in a,\ (y, z) \in b\}.$$

The converse relation of $a\colon A \to B$ is obtained by reversing pairs, and written as $a^\sharp\colon B \to A$. This involution is regular in the sense of von Neumann, i.e. $aa^\sharp a = a$, for all relations a. As a consequence, a *monorelation*, i.e. a monomorphism in the category RelAb, is characterised by the condition $a^\sharp a = 1$; an *epirelation* by the dual condition $aa^\sharp = 1$.

The category Ab is embedded in its category of relations, identifying a homomorphism with its graph. This embedding preserves monomorphisms and epimorphisms (but we shall see that a monorelation is more general than an injective homomorphism). *Isomorphisms are the same*, in these categories.

Let us come back to the bicartesian square making S into a subquotient M/N of the abelian group A

$$\begin{tikzcd} M \arrow[r, tail, "m"] \arrow[d, "p"'] & A \arrow[d, "q"] \\ S \arrow[r, tail, "n"'] \arrow[ur, dashed, "s"] & A/N \end{tikzcd} \tag{5}$$

This bicartesian square determines *one relation* $s = mp^\sharp = q^\sharp n\colon S \to A$, that sends the class $[x] \in M/N$ to all the elements of the lateral $x + N$ in A. It is actually a monorelation (since $s^\sharp s = \mathrm{id}(S)$) and all monorelations with values in A are of this type, up to isomorphism. The *subquotients* of the abelian group A amount thus to the *subobjects of A in* RelAb.

RelAb makes possible to consider a more general notion of induction on subquotients, as in [M1]. Given a relation $a\colon A \to B$, a subquotient $s\colon S \to A$ of its domain and a subquotient $t\colon T \to B$ of its codomain, we say that *a induces from s to t* the relation

$$t^\sharp as\colon S \to T. \tag{6}$$

If a is a homomorphism and has a regular induction from S to T, the regularly induced homomorphism $S \to T$ considered above coincides with $t^\sharp as$.

If s, t are subquotients of the same object A, the relation $u = t^\sharp s\colon S \to T$ induced by the identity of A will be called the *canonical relation* from s to t; and a *canonical isomorphism* if it is an isomorphism (of RelAb or Ab, equivalently). The isomorphism u need not be regularly induced.

Writing the subquotient s as H/K, and t as H'/K', it is easy to verify the following properties of the canonical relation $u = t^\sharp s\colon H/K \to H'/K'$:

(a)	u is everywhere defined	$\Leftrightarrow$	$H \subset H' \vee K$,
(a*)	u has total values	$\Leftrightarrow$	$H' \subset H \vee K'$,
(b)	u has a null annihilator	$\Leftrightarrow$	$H \wedge K' \subset K$,
(b*)	u has a null indeterminacy	$\Leftrightarrow$	$H' \wedge K \subset K'$,
(c)	u is an isomorphism	$\Leftrightarrow$	$(H \vee K' = H' \vee K,\ \ H \wedge K' = H' \wedge K)$.

It follows that

(d) u is a regularly induced isomorphism $\Leftrightarrow$ $(K = H \wedge K',\ \ H' = H \vee K')$,

which shows that a regularly induced isomorphism is the same as a second-type Noether isomorphism

$$H/(H \wedge K') \to (H \vee K')/K'. \tag{7}$$

We write $H/K \, \Phi \, H'/K'$ the property expressed in (c). It is obviously reflexive and symmetric, but *not transitive in general*, as we show below.

It is easy to see that, if $H/K \, \Phi \, H'/K'$, there is a commutative diagram of canonical isomorphisms (between Φ-related subquotients of A)

$$\begin{tikzcd}
& (H \vee H')/(K \vee K') & \\
H/K \arrow[ur] \arrow[rr, dashed, "u"] & & H'/K' \arrow[ul] \\
& (H \wedge H')/(K \wedge K') \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{8}$$

where the solid arrows are *regularly induced* (Noether) isomorphisms; this is important, because regular induction is always respected by composition.

3. A case of incoherence

The following examples show some instances of *inconsistency of induction on subquotients*: first, canonical isomorphisms need not be closed under composition; second, if we extend them in this sense the result need not be uniquely determined.

As in Mac Lane's book [M1], our examples of inconsistency are based on the lattice $L(A)$ of subgroups of $A = \mathbb{Z} \oplus \mathbb{Z}$, and more particularly on the (*non-distributive*) triple formed of the diagonal Δ and two of its complements, the subgroups A_1 and A_2

$$\begin{array}{ll} A_1 = \mathbb{Z} \oplus 0, & A_2 = 0 \oplus \mathbb{Z}, \\ A_i \vee \Delta = A, & A_i \wedge \Delta = 0. \end{array} \tag{9}$$

We thus have the subquotients $m_i\colon A_i \to A$ and $s = p^\sharp\colon A/\Delta \to A$.

(a) The identity of A induces two canonical isomorphisms $u_i = pm_i\colon A_i \to A/\Delta$ (regularly induced Noether isomorphisms, by (9)), and a canonical isomorphism $u_2^{-1}\colon A/\Delta \to A_2$ (that is *not* regularly induced).

Then, the composed isomorphism $w = u_2^{-1}u_1\colon A_1 \to A_2$ *is not canonical.* Indeed:

$$w\colon A_1 \to A/\Delta \to A_2, \qquad (x,0) \mapsto [(x,0)] = [(0,-x)] \mapsto (0,-x), \tag{10}$$

while the canonical relation $m_2^\sharp.m_1\colon A_1 \to A_2$ has graph $\{(0,0)\}$.

(b) Using the subgroup $\Delta' = \{(x,-x) \mid x \in \mathbb{Z}\}$ instead of the diagonal Δ, we get the *opposite* composed isomorphism from A_1 to A_2

$$A_1 \to A/\Delta' \to A_2 \qquad (x,0) \mapsto [(x,0)] = [(0,x)] \mapsto (0,x). \tag{11}$$

This shows that a composite $A_1 \to A_2$ of canonical isomorphisms between subquotients of $\mathbb{Z} \oplus \mathbb{Z}$ *is not determined.*

Now, a change of sign can be quite important, in homological algebra and algebraic topology. For instance, it is the case in the usual argument proving that 'even-dimensional spheres cannot be combed': if the sphere $\mathbb{S}^n$ has a non-null vector field, then its antipodal map $t\colon \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to the identity, and the degree $(-1)^{n+1}$ of t must be 1. The conclusion cannot be obtained if we only know the induced homomorphism $t_{*n}\colon H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$ up to sign change.

4. Coherent systems of isomorphisms

Let X be a sublattice of the (modular) lattice $L(A)$ of subgroups of the abelian group A. We are interested in the set $\hat{X}$ of all the subquotients of A with numerator and denominator in X, whose coherence is discussed below.

Plainly, the set $\hat{X}$ can be identified with the set X_2 of decreasing pairs (numerator, denominator) of X, where the relation $(x,y)\,\Phi\,(x',y')$ is expressed by the following two equivalent conditions:

(a) $\quad x \vee y' = x' \vee y, \qquad\qquad x \wedge y' = x' \wedge y,$

(b) $\quad x \leqslant x' \vee y, \quad x' \leqslant x \vee y', \qquad x \wedge y' \leqslant y, \quad x' \wedge y \leqslant y'.$

For a system Σ of subquotients of A (usually of the previous form), we are interested in the following equivalent properties

(i) whenever $u\colon S \to S'$ and $v\colon S' \to S''$ are induced isomorphisms between elements of the system, the composed isomorphism vu coincides with the canonical relation $S \to S''$,

(ii) the relation Φ is an equivalence relation among the subquotients of Σ, and all diagrams of canonical isomorphisms between them commute.

When this holds we say that Σ is a *coherent system of subquotients* of A. We also express this fact saying that the canonical isomorphisms among all $S \in \Sigma$ *are closed under composition*, or *form a coherent system of isomorphisms.* (Since Mac Lane's paper

[M2], a coherence theorem in category theory states that, under suitable assumptions, all the diagrams of a given type commute.)

When such a system has been fixed (e.g. using the Coherence theorem below) we shall express the (equivalence!) relation $S\,\Phi\,S'$ of Σ by saying that the subquotients S and S' are *canonically isomorphic* (within Σ). But expressing in this way the relation Φ when transitivity does not hold is misleading and should be carefully avoided.

We have seen that the whole system of subquotients of $\mathbb{Z}^2$ is not coherent (Section 3); the same holds for any object $A \oplus A$, where A is any non trivial object of an abelian category.

Even when a set $\Sigma = \hat{X}$ is coherent, one should not expect that all the induced *homo*morphisms (or even less relations) be closed under composition. In fact, the composite of the canonical homomorphisms

$$A/0 \to A/A \to 0/0 \to A/0$$

is null, while the canonical homomorphism $A/0 \to A/0$ is the identity (and all these subquotients necessarily belong to $\hat{X}$).

5. Lemma

Let X be a modular lattice. The following conditions are equivalent:

(i) the lattice X is distributive,

(ii) the relation $(x, y)\,\Phi\,(x', y')$ defined above on the set X_2 of decreasing pairs of X is an equivalence relation.

Proof. Let X be distributive, and assume that $(x, y)\,\Phi\,(x', y')\,\Phi\,(x'', y'')$. Then:

$$x = (x' \vee y) \wedge x = (x'' \vee y' \vee y) \wedge x \;\leqslant\; x'' \vee (y' \wedge x) \vee y = x'' \vee (y \wedge x') \vee y = x'' \vee y.$$

The other three inequalities of $(x, y)\,\Phi\,(x'', y'')$, in form (b) of Section 4, are proved in a similar way.

Conversely, suppose that the relation Φ is transitive. Let $M = \{m', x, y, z, m''\}$ be a quasi sublattice of X, where the meet (resp. join) of any two elements out of x, y, z is m' (resp. m''). Then we have $(x, m')\,\Phi\,(m'', y)\,\Phi\,(z, m')$, whence $(x, m')\,\Phi\,(z, m')$ and $x = z$.

In other words, the modular lattice X cannot have a quasi sublattice M as above, formed of five *distinct* elements; by a well-kown theorem of lattice theory ([Bi], II.8, Theorem 13), the lattice X must be distributive. ■

6. Coherence Theorem of Homological Algebra (Reduced form)

Let X be a sublattice of the lattice $L(A)$ of subgroups of the abelian group A. Then the following conditions are equivalent:

(i) the lattice X is distributive,

(ii) the family $\hat{X}$ is coherent (i.e. the canonical isomorphisms among subquotients of A with numerator and denominator belonging to X are closed under composition).

Proof. If (ii) holds, the relation Φ is transitive in $\hat{X}$ (or equivalently in X_2) and X is distributive, by the previous lemma.

Conversely, let us assume that X is distributive, and consider two canonical isomorphisms between three subquotients

$$u\colon H/K \to H'/K', \qquad v\colon H'/K' \to H''/K''. \tag{12}$$

We must prove that the composite vu is the canonical relation $w\colon H/K \to H''/K''$. By Lemma 5, these three subquotients are Φ-equivalent.

Let us write

$$H_0 = H \wedge H' \wedge H'', \qquad K_0 = K \wedge K' \wedge K''.$$

By Section 2, we can form the following commutative diagram, where all subquotients are Φ-equivalent, and the solid arrows are *regularly* induced by $\mathrm{id}(A)$

$$\begin{tikzcd}
H/K \arrow[rr, dashed, "u"] & & H'/K' \arrow[rr, dashed, "v"] & & H''/K'' \\
(H \wedge H')/(K \wedge K') \arrow[u] \arrow[urr] & & & & (H' \wedge H'')/(K' \wedge K'') \arrow[u] \arrow[ull] \\
& & H_0/K_0 \arrow[ull] \arrow[urr] & &
\end{tikzcd} \tag{13}$$

But we can also form a second commutative diagram, where the solid arrows are again regularly induced

$$\begin{tikzcd}
H/K \arrow[rr, dashed, "w"] & & H''/K'' \\
(H \wedge H')/(K \wedge K') \arrow[u] & (H \wedge H'')/(K \wedge K'') \arrow[ul] \arrow[ur] & (H' \wedge H'')/(K' \wedge K'') \arrow[u] \\
& H_0/K_0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd} \tag{14}$$

Since the four solid arrows of the 'boundary' of these two diagrams coincide, the thesis follows: $vu = w$.

■

7. Filtered chain complexes

Let us now consider one of the most usual structures giving rise to a spectral sequence, a *filtered chain complex* A_* of abelian groups, with (canonically) *bounded* filtration

$$A_* = ((A_n), (\partial_n), (F_p A_n)). \tag{15}$$

This is a chain complex of abelian groups

$$\dots\ A_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \ \dots \ \longrightarrow A_1 \xrightarrow{\partial_1} A_0 \qquad\qquad (\partial_n\partial_{n+1}=0),$$

where each component A_n has a filtration of length $n{+}1$, consistently with the differentials:

$$\begin{gathered} 0 \subset F_0A_n \subset \dots \subset F_pA_n \subset \dots \subset F_nA_n = A_n, \\ \partial_{n+1}(F_pA_{n+1}) \subset F_pA_n. \end{gathered} \tag{16}$$

On each component A_n the filtrations of A_{n+1} and A_{n-1} produce a second finite filtration (of length $2n+3$), by direct and inverse images along the differentials (while the other components have a trivial effect)

$$\begin{aligned} 0 \subset \partial_{n+1}(F_0A_{n+1}) \subset \dots \subset \partial_{n+1}(F_{n+1}A_{n+1}) = \operatorname{Im}\partial_{n+1} \\ \subset \operatorname{Ker}\partial_n \subset \partial_n^{-1}(F_0A_{n-1}) \subset \dots \subset \partial_n^{-1}(F_{n-1}A_{n-1}) = A_n. \end{aligned} \tag{17}$$

By a well-known Birkhoff theorem on the free modular lattice generated by two chains ([Bi], III.7, Theorem 9), the two filtrations generate a finite, *distributive* lattice of subgroups of A_n, that can be represented as (a quotient of) a lattice of subsets of the plane. (Notice the crucial role played here by the condition $\partial\partial = 0$: without that, the lattice generated by the data would not be distributive.)

In particular, we are interested in the following subobjects of A_n, forming a filtration of F_pA_n that is the 'trace' of the second filtration (17) (as usual, we let $n = p + q$):

$$\begin{aligned} Z^r_{pq}(A_*) &= F_pA_n \wedge \partial^{-1}(F_{p-r}A_{n-1}) \qquad &(\textit{relative cycles}), \\ B^r_{pq}(A_*) &= F_pA_n \wedge \partial(F_{p+r}A_{n+1}) \qquad &(\textit{relative boundaries}). \end{aligned} \tag{18}$$

Now, the term E^r_{pq}, of the spectral sequence of A_* is usually defined as a subquotient of A_n (with $n = p + q$), by one of the following 'equivalent' formulas:

$$E^r_{pq}(A_*) = Z^r_{pq}/(Z^{r-1}_{p-1,q+1} \vee B^{r-1}_{pq}), \tag{19}$$

$$E^r_{pq}(A_*) = (Z^r_{pq} \vee F_{p-1}A_n)/(B^{r-1}_{pq} \vee F_{p-1}A_n), \tag{20}$$

that are linked by a canonical isomorphism, regularly induced from the first to the second subquotient.

The first expression is used, for instance, in Hilton - Wylie [HiW], Section 10.3, and Spanier [Sp], 9.1. The second is used in Mac Lane's 'Homology' [M1], XI.3. Weibel [We] uses both, in Section 5.4 (with a different notation).

And indeed, no problem can *here* arise from using different formulas linked by canonical isomorphisms, *because of the distributivity of the system.*

8. A non-distributive structure: the bifiltered complex

We now construct a *bifiltered* chain complex C where the terms $E^2_{10}(C')$ and $E^2_{10}(C'')$ pertaining to the two filtrations involve the abelian group $A = \mathbb{Z} \oplus \mathbb{Z}$ and the non-distributive triple of subgroups already used in Section 3: $A_1 = \mathbb{Z} \oplus 0$, $A_2 = 0 \oplus \mathbb{Z}$ and Δ (the diagonal).

Let us start from the filtered chain complex

$$C' = ((C_n), (\partial_n), (F'_p C_n)),$$

having only two non-trivial components, in degree 0 and 1

$$\begin{aligned} C_0 &= \mathbb{Z}, & C_1 &= A = \mathbb{Z} \oplus \mathbb{Z}, \\ \partial_1 \colon C_1 &\to C_0, & \partial_1(x, y) &= x - y, \end{aligned}$$

and the following filtration F':

$$F'_0 C_0 = C_0, \qquad F'_0 C_1 = A_1, \quad F'_1 C_1 = C_1. \tag{21}$$

Using the formulas (19) and (20), the term $E^2_{10}(C')$ can be computed with the following canonically isomorphic subquotients of C_1

$$\begin{gathered} H/(H \wedge K) \;\cong\; (H \vee K)/K, \\ H = F'_1 C_1 \wedge \operatorname{Ker} \partial_1, \qquad K = F'_0 C_1. \end{gathered} \tag{22}$$

Here we get $H = \Delta$, $K = A_1$, which gives for $E^2_{10}(C')$ the isomorphic subquotients $\Delta \;\cong\; A/A_1$ of the component $C_1 = A = \mathbb{Z} \oplus \mathbb{Z}$.

Replacing the filtration F' with F'', where $F''_0 C_1 = A_2$ and the rest is unchanged, we get a second filtered chain complex C''. The term $E^2_{10}(C'')$ is computed as above, with $H = \Delta$ and $K = A_2$; we get now: $\Delta \;\cong\; A/A_2$.

Staying inside *one* of the filtered complexes C' or C'', the isomorphism (22) does not lead to any problem, because of the previous coherence theorem. But if we consider the bifiltered chain complex C, equipped with *both filtrations*

$$C = ((C_n), (\partial_n), (F'_p C_n), (F''_p C_n)), \tag{23}$$

we get two spectral sequences, in a *non*-distributive framework where the coherence theorem does not apply.

Now, the first formula of (22) seems to tell us that the terms $E^2_{10}(C')$, $E^2_{10}(C'')$ of these two systems are precisely the same thing, namely the subgroup Δ of C_1. On the other hand, the formula $(H \vee K)/K$ gives $\mathbb{Z}^2/A_1$ or $\mathbb{Z}^2/A_2$, according to the case. These are isomorphic groups, of course, but *quite different quotients* of the component C_1. We do not even know how to define the relation $E^2_{10}(C') \to E^2_{10}(C'')$ induced by the identity of C: the first formula would give the identity, while the second would give the 'chaotic' relation, of graph $E^2_{10}(C') \oplus E^2_{10}(C'')$.

Therefore, for a bifiltered chain complex *there is no consistent theory involving both spectral sequences.* Errors are normally avoided because the only interrelation of the two systems which is used in practice is the fact that they both converge to the homology of C; yet the problem should not be ignored.

9. The double complex

This point can be further investigated dealing with double complexes - a distributive system where a consistent theory of two interacting spectral sequences exists, provided *one does not define them in the usual way*, namely via the associated total complex and its bifiltration.

In fact, let A_* be a *double complex* of abelian groups

$$A_* = ((A_{pq}), (\partial'_{pq}), (\partial''_{pq})). \tag{24}$$

(A_{pq}) is a family of groups, indexed on $\mathbb{N} \times \mathbb{N}$. The differentials

$$\partial'_{pq} \colon A_{pq} \to A_{p-1,q}, \qquad \partial''_{pq} \colon A_{pq} \to A_{p,q-1},$$

are assumed to commute (i.e. to form a commutative diagram), and of course to have null composites $A_{pq} \to A_{p-2,q}$ and $A_{pq} \to A_{p,q-2}$.

The usual procedure is to construct the associated *total complex* $C = ((C_n), (\partial_n))$

$$\begin{aligned} C_n &= \textstyle\bigoplus_{p=0,\dots n} A_{p,n-p}, \qquad u_{pq} \colon A_{pq} \to C_{p+q}, \\ \partial_n(x) &= \textstyle\sum_{p+q=n} u_{p-1,q}\, \partial'_{pq}(x) + (-1)^p\, u_{p,q-1}\, \partial''_{pq}(x) \qquad (x \in A_{pq}). \end{aligned} \tag{25}$$

C has two (canonically bounded) filtrations

$$F'_p C_n = \textstyle\bigoplus_{r=0,\dots p} A_{r,n-r}, \qquad F''_q C_n = \textstyle\bigoplus_{r=0,\dots q} A_{n-r,r}, \tag{26}$$

and each of them determines a spectral sequence.

But this is *a non-distributive system whose dangers of inconsistency have already been highlighted above.*

Consider the following double complex of abelian groups, with only three non trivial components

$$A_{00} = A_{01} = A_{10} = \mathbb{Z}, \qquad \partial''_{01} = -\partial'_{10} = \mathrm{id}(\mathbb{Z}). \tag{27}$$

The associated total complex C is precisely the bifiltered chain complex of Section 8, with

$$C_0 = \mathbb{Z}, \qquad C_1 = A_{01} \oplus A_{10} = \mathbb{Z} \oplus \mathbb{Z},$$
$$\partial_1(x, y) = \partial''_{01}(x) + \partial'_{10}(y) = x - y,$$

and the filtrations F', F'' described there.

Computing the previous terms on C, as $E^2_{10}(C')$ and $E^2_{10}(C'')$, *one gets subquotients of* C_1 *whose interaction depends on the formulas we are using*, as we have seen above.

Nevertheless, we can introduce the two spectral sequences of A_* *without using direct sums and the total complex*, via the following formulas (cf. [G1], p. 280) that define $'E^r_{pq}$ and $''E^r_{pq}$ *as subquotients of* A_{pq} (not of C_{p+q})

$$\begin{aligned} 'E^r_{pq} &= (((\partial'^{*}\partial''_{*})^{r-1}(A) \wedge \partial''^{*}(0)) \,/\, ((\partial'_{*}\partial''^{*})^{r-1}(0) \vee \partial''_{*}(A)))_{pq}, \\ ''E^r_{pq} &= (((\partial''^{*}\partial'_{*})^{r-1}(A) \wedge \partial'^{*}(0)) \,/\, ((\partial''_{*}\partial'^{*})^{r-1}(0) \vee \partial'_{*}(A)))_{pq}. \end{aligned} \tag{28}$$

(Here, the direct and inverse images of subgroups along a homomorphism $f\colon X \to Y$ are denoted as $f_*\colon L(X) \to L(Y)$ and $f^*\colon L(Y) \to L(X)$, respectively.)

In the abelian case, as proved in [G1], these formulas yield the usual terms up to canonical isomorphism (even though - plainly - there is no coherent system containing all these subquotients).

10. The full coherence theorem

We end with mentioning (without proof) a more complete form of our coherence theorem.

As recalled in the Introduction, it was proved by the author in a series of papers of the 1970's ([G2] - [G8]). A self-contained proof will be given in a book in preparation [G13], with a statement including many other equivalent conditions, and various applications to the theory of spectral sequences (some of them already published in [G9] - [G13]).

The required setting is an extension of abelian categories. A *p-exact* category, i.e. an exact category in the sense of Puppe and Mitchell (see [Pu, Mt, T1, T2, Br, BrP, HeS, AHS, FS]), is a category with a zero object, where every map factorises as a cokernel (of some morphism) followed by a kernel. The setting is selfdual, and the existence of cartesian products is not assumed - in contrast with Barr-exact categories [Ba].

A p-exact category E is said to be *distributive* if all its lattices of subobjects are distributive. The main example is the category $\mathcal{I}$ of sets and partial bijections, where every small distributive p-exact category can be exactly embedded.

Coherence theorem of homological algebra

For a p-exact category E*, the following conditions are equivalent:*

(i) canonical isomorphisms between subquotients of the same object are closed under composition,

(ii) induced isomorphisms between subquotients (induced by arbitrary homomorphisms, or even by relations) are preserved by composition,

(iii) E *is* distributive*,*

(iv) the category of relations Rel$\mathsf{E}$ *is* orthodox *(i.e. its idempotent endomorphisms are closed under composition).*

Notice that a non-trivial abelian category cannot be distributive: a lattice of subobjects $L(A \oplus A)$ is not distributive, unless $A = 0$. However, every p-exact category (including the abelian ones!) has a *distributive expansion* Dst$\mathsf{E}$, a distributive p-exact category whose objects are the pairs (A, X) consisting of an object A of E and a distributive sublattice $X \subset L(A)$.

Combining (part of) the coherence theorem with the distributive expansion, we get the restricted form proved above, in Section 6.

Let us also remark that the coherence theorem draws new, tight links between:

- homological algebra,

- lattice theory,
- the theory of regular, orthodox, quasi-inverse and inverse semigroups.

The latter was developed from the 1960's by B.M. Schein [Sc], N.R. Reilly and M.E. Scheiblich [ReS], T.E. Hall [Ha], M. Yamada [Ya], A.H. Clifford and G.B. Preston [CP], and others. See also J.M. Howie's book [Ho]. These notions have been extended to categories in [G2] - [G5].

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