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Hardy type spaces on certain noncompact manifolds and applications

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Abstract

In this paper we consider a complete connected noncompact Riemannian manifold M with Ricci curvature bounded from below, positive injectivity radius and spectral gap b. We introduce a sequence $X^1(M), X^2(M), \ldots$ of new Hardy spaces on M, the sequence $Y^1(M), Y^2(M), \ldots$ of their dual spaces, and show that these spaces may be used to obtain endpoint estimates for purely imaginary powers of the Laplace–Beltrami operator and for more general spectral multipliers associated to the Laplace–Beltrami operator $\mathcal{L}$ on M. Under the additional condition that the volume of the geodesic balls of radius r is controlled by $C\, r^\alpha\, \mathrm{e}^{2\sqrt{b}r}$ for some real number α and for all large r, we prove also an endpoint result for first order Riesz transforms $\nabla\mathcal{L}^{-1/2}$.

In particular, these results apply to Riemannian symmetric spaces of the noncompact type.

1. *Introduction*

The Riesz transform $\nabla(-\Delta)^{-1/2}$ and the purely imaginary powers $(-\Delta)^{iu}$, u in $\mathbb{R}$, of the Laplacian Δ are prototypes of singular integral operators on $\mathbb{R}^n$. They are bounded on $L^p(\mathbb{R}^n)$ for all p in $(1,\infty)$, and unbounded on $L^1(\mathbb{R}^n)$ and on $L^\infty(\mathbb{R}^n)$ [**38**]. Classical results (see the seminal papers [**24**, **18**]) state that singular integral operators satysfying the so called Hörmander integral condition are of weak type 1 and bounded from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. These results apply, in particular, to $\nabla(-\Delta)^{-1/2}$ and $(-\Delta)^{iu}$. One reason to choose $(-\Delta)^{iu}$ as an example of singular integral operators is that it plays a fundamental role in the functional calculus for $-\Delta$, for functions of the Laplacian may, at least formally, be reconstructed from $(-\Delta)^{iu}$ via a subordination formula involving the Mellin transform (see the fundamental works [**37**, **14**]).

Now suppose that M is a Riemannian manifold with Riemannian measure μ, and denote by $-\mathcal{L}$ and ∇ the associated Laplace–Beltrami operator and covariant derivative respectively. It is natural to speculate whether the analogues of the aforementioned results hold for the operators $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{iu}$. The multiplier result for generators of semigroups proved in [**37**, **14**] applies to $\mathcal{L}^{iu}$ and gives the $L^p(M)$ boundedness of these operators for p in $(1,\infty)$. The $L^p(M)$ boundedness of $\nabla\mathcal{L}^{-1/2}$ for p in $(1,2)$, and without additional assumptions on M, seems to be a challenging problem, and it is the object of a very active line of research (see, for instance, [**13**, **4**] and the references therein).

As far as endpoint estimates for $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{iu}$ are concerned, interesting results have been obtained in the case where μ is doubling and M satisfies some extra assumptions, such as appropriate on-diagonal estimate for the heat kernel [**13**], or scaled Poincaré inequality [**36**, **29**,

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5]. Note that when μ is doubling, M is a space of homogeneous type in the sense of Coifman and Weiss, and a well known theory of atomic Hardy spaces is available [**12**].

In this paper we consider a complete connected noncompact Riemannian manifold M with Ricci curvature bounded from below, positive injectivity radius and strictly positive bottom b of the spectrum of $\mathcal{L}$. It may be worth observing that under these assumptions the Riemannian measure is nondoubling and that the volume of geodesic balls in M grow exponentially with the radius. Recall that for a Riemannian manifold satisfying the above assumptions there are positive constants α, β and C such that

$$\mu\big(B(p,r)\big) \leq C\, r^{\alpha}\, \mathrm{e}^{2\beta\, r} \qquad \forall r \in [1,\infty) \quad \forall p \in M, \tag{1.1}$$

where $\mu\big(B(p,r)\big)$ denotes the Riemannian volume of the geodesic ball with centre p and radius r. Notable examples of such manifolds are nonamenable connected unimodular Lie groups equipped with a left invariant Riemannian distance, and symmetric spaces of the noncompact type with the Killing metric.

In this setting, weak type 1 estimates for $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{iu}$ are known only when M is a Riemannian symmetric space of the noncompact type [**1**, **2**, **26**, **27**, **35**].

Manifolds satisfying the above assumptions fall into the class of measured metric spaces X considered in [**7**], where the authors, following up earlier works of A.D. Ionescu [**25**] and of E. Russ [**36**], defined an atomic Hardy space $H^1(X)$ and a space of functions of bounded mean oscillation $BMO(X)$. Both $H^1(X)$ and $BMO(X)$ are defined much as in the classical case of spaces of homogeneous type, the only difference being that atoms in the definition of $H^1(X)$ are supported in balls with radius at most 1, and that in the definition of $BMO(X)$ averages are taken only on balls of radius at most 1. As a consequence, they proved that if $\mathcal{T}$ is bounded on $L^2(X)$ and its kernel $k_{\mathcal{T}}$ satisfies the following local Hörmander's type condition

$$\sup_{B\in\mathcal{B}_1}\, \sup_{y\in B} \int_{(2B)^c} |k_{\mathcal{T}}(x,y) - k_{\mathcal{T}}(x,c_B)| \,\mathrm{d}\mu(x) < \infty, \tag{1.2}$$

where $\mathcal{B}_1$ denotes the collection of all balls in X of radius at most 1, then $\mathcal{T}$ is bounded on $L^p(X)$ for all p in $(1,2]$ and from the atomic Hardy space $H^1(X)$ to $L^1(X)$.

The starting point of our work is the perhaps surprising fact that when $\mathcal{L}$ is the Laplace–Beltrami operator associated to the Killing metric on Riemannian symmetric spaces of the noncompact type the operators $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{iu}$, $u \neq 0$, are unbounded operators from $H^1(M)$ to $L^1(M)$. The proof of this fact hinges on quite delicate estimates of the inverse spherical Fourier transform of the associated multiplier, and will appear in [**31**]. Note that, as a consequence, their Schwartz kernels $k_{\mathcal{L}^{iu}}$ and $k_{\nabla\mathcal{L}^{-1/2}}$ do not satisfy (1.2).

The purpose of this paper is to introduce a sequence $X^1(M), X^2(M), \ldots$ of new spaces of Hardy type on M, and the sequence $Y^1(M), Y^2(M), \ldots$ of their dual spaces, and show that these spaces may be used to obtain endpoint estimates for $\nabla\mathcal{L}^{-1/2}$, $\mathcal{L}^{iu}$, and for more general spectral multipliers of $\mathcal{L}$. The space $X^k(M)$ is defined as follows. Denote by $\mathcal{U}_{\beta^2}$ the operator $\mathcal{L}\,(\beta^2\mathcal{I} + \mathcal{L})^{-1}$. It is straightforward to check that $\mathcal{U}_{\beta^2}$ is a bounded injective operator on $L^1(M) + L^2(M)$. Denote by $X^k(M)$ the range of the restriction of $\mathcal{U}_{\beta^2}^k$ to $H^1(M)$, endowed with the norm

$$\|f\|_{X^k} = \|\mathcal{U}_{\beta^2}^{-k} f\|_{H^1}.$$

By definition, each arrow of the following commutative diagram is an isometric isomorphism of Banach spaces.

$$\begin{tikzcd}
H^1(M) \arrow[d, "\mathcal{U}_{\beta^2}"'] \arrow[dr, "\mathcal{U}^2_{\beta^2}"'] \arrow[drr, "\mathcal{U}^3_{\beta^2}"'] \arrow[drrr, "\mathcal{U}^4_{\beta^2}"'] & & & \\
X^1(M) \arrow[r, "\mathcal{U}_{\beta^2}"'] & X^2(M) \arrow[r, "\mathcal{U}_{\beta^2}"'] & X^3(M) \arrow[r, "\mathcal{U}_{\beta^2}"'] & \cdots
\end{tikzcd}$$

Thus, $X^k(M)$ is an isometric copy of $H^1(M)$ for each positive integer k. Furthermore, we shall prove (see Section 5) that

$$H^1(M) \supset X^1(M) \supset X^2(M) \supset \cdots,$$

with proper inclusions. These spaces have nice interpolation properties; for each positive integer k, and for every p in $(1,2)$, $L^p(M)$ is an interpolation space between $X^k(M)$ and $L^2(M)$ by the complex method (see Section 2).

The main results of this paper are contained in Section 4, and justify, *a posteriori*, the introduction of the spaces $X^k(M)$. In particular, Theorem 4.3 states that if m is a holomorphic function in the strip $\mathbf{S}_\beta = \{\zeta \in \mathbb{C} : \mathrm{Im}(\zeta) \in (-\beta, \beta)\}$ that satisfies

$$|D^j m(\zeta)| \leq C \max\big(|\zeta^2 + \beta^2|^{-\tau-j}, |\zeta|^{-j}\big) \qquad \forall \zeta \in \mathbf{S}_\beta \quad \forall j \in \{0,1,\ldots,J\}, \tag{1.3}$$

for some nonnegative τ and for a sufficiently large integer J, then $m\big(\sqrt{\mathcal{L} - b}\big)$ is bounded from $H^1(M)$ to $L^1(M)$ and from $L^\infty(M)$ to $BMO(M)$ in the case where $b < \beta^2$ and from $X^k(M)$ to $H^1(M)$ and from $BMO(M)$ to $Y^k(M)$ in the case where $b = \beta^2$ and $k > \tau + J$. This provides, in the case where $b = \beta^2$, endpoint estimates for operators of the form $\mathcal{L}^{iu}$ (when $\tau = 0$), but also for "more singular operators", such as $\mathcal{L}^{iu-\tau}\,(I + \mathcal{L})^\tau$, whose kernels have a comparatively slow decay at infinity. We shall call *strongly singular* all the multipliers satisfying (1.3). Strongly singular spectral multipliers were first introduced in [**35**], where the authors showed that they satisfy weak type 1 estimates when M is a Riemannian noncompact symmetric spaces. We remark that the methods of [**35**] hinge on quite precise estimates of the kernel of these operators, obtained by using the inversion formula for the spherical Fourier transform. Weak type 1 estimates for such operators seem out of reach in the more general setting of this paper. Note that strongly singular multipliers may have a rather singular behaviour near the points $\pm i\beta$, and still satisfy an endpoint result for $p = 1$. We emphasise that this is in sharp constrast with the Euclidean case, where such a phenomenon cannot occur.

We give applications also to first order Riesz transforms. It follows from work of T. Coulhon and X.T. Duong [**13**] that, in our setting, the first order Riesz transform $\nabla\mathcal{L}^{-1/2}$ is bounded on $L^p(M)$ for all p in $(1,2]$ and that the translated Riesz transform $\nabla(\mathcal{I} + \mathcal{L})^{-1/2}$ is of weak type 1. Russ complemented this result by showing that $\nabla(\mathcal{I} + \mathcal{L})^{-1/2}$ map $H^1(M)$ into $L^1(M)$. Observe that if we consider the part off the diagonal of the kernel of $\nabla(\mathcal{I} + \mathcal{L})^{-1/2}$, then the corresponding integral operator is bounded on $L^1(M)$. This is no longer true for the kernel of the Riesz transform $\nabla\mathcal{L}^{-1/2}$, which decays much slower at infinity. Despite this, we prove that if $b = \beta^2$, then $\nabla\mathcal{L}^{-1/2}$ is bounded from $X^k(M)$ to $L^1(M)$ for large k. Applications of these spaces to higher order Riesz transforms associated to the Laplace–Beltrami operator on noncompact symmetric spaces and to multipliers for the spherical Fourier transform will be considered in a forthcoming paper [**31**].

The space $X^k(M)$ admits an interesting characterisation in terms of atoms in $H^1(M)$ that satisfy infinitely many cancellation conditions. Its proof, which is rather long, is deferred to a forthcoming paper [**32**].

We now briefly outline the content of the paper. In the next section we define the new Hardy spaces $X^k(M)$ and their duals $Y^k(M)$ in the fairly general framework of the measured metric spaces considered in [**7**] and show that they have natural interpolation properties. In Section 3 we specialise to Riemannian manifolds with Ricci curvature bounded from below, positive injectivity radius and strictly positive bottom of the spectrum and we prove some further properties of the new Hardy spaces in this setting. We also state a theorem on the boundedness on $H^1(M)$ of functions of the Laplacian (Theorem 3.4), which is of independent interest and plays a crucial role in the proof of the main results of this paper. The proof of this theorem is deferred to Section 5. The main results of the paper, i.e. the endpoint estimates for strongly singular multipliers and for the Riesz transform are stated and proved in Section 4.

We will use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards. If $\mathcal{T}$ is a bounded linear operator from the Banach space A to the Banach space B, we shall denote by $|||\mathcal{T}|||_{A;B}$ its norm. If $A = B$ we shall simply write $|||\mathcal{T}|||_{A}$ instead of $|||\mathcal{T}|||_{A;A}$.

2. *New Hardy spaces on metric spaces and interpolation*

Suppose that (M, d, μ) is a measured metric space, and denote by $\mathcal{B}$ the family of all balls on M. We assume that $\mu(M) > 0$ and that every ball has finite measure. For each B in $\mathcal{B}$ we denote by c_B and r_B the centre and the radius of B respectively. Furthermore, we denote by $c\,B$ the ball with centre c_B and radius $c\,r_B$. For each *scale parameter* s in $\mathbb{R}^+$, we denote by $\mathcal{B}_s$ the family of all balls B in $\mathcal{B}$ such that $r_B \leq s$.

Basic assumptions 2.1. We assume throughout that M is *unbounded* and possesses the following properties:

(i) *local doubling property* (LD): for every s in $\mathbb{R}^+$ there exists a constant D_s such that

$$\mu(2B) \leq D_s\, \mu(B) \qquad \forall B \in \mathcal{B}_s; \tag{2.1}$$

(ii) *isoperimetric property* (I): there exist κ_0 and C in $\mathbb{R}^+$ such that for every bounded open set A

$$\mu\Big(\big\{x \in A : d(x, A^c) \leq \kappa\big\}\Big) \geq C\,\kappa\,\mu(A) \qquad \forall \kappa \in (0, \kappa_0];$$

(iii) *approximate midpoint property* (AM): there exist R_0 in $[0, \infty)$ and γ in $(1/2, 1)$ such that for every pair of points x and y in M with $d(x, y) > R_0$ there exists a point z in M such that $d(x, z) < \gamma\, d(x, y)$ and $d(y, z) < \gamma\, d(x, y)$;

(iv) there is a *semigroup of linear operators* $\{\mathcal{H}^t\}$ acting on $L^1(M) + L^2(M)$ such that

(a) the restriction of $\{\mathcal{H}^t\}$ to $L^1(M)$ is a strongly continuous semigroup of contractions;

(b) the restriction of $\{\mathcal{H}^t\}$ to $L^2(M)$ is strongly continuous, and has *spectral gap* $b > 0$, i.e.

$$\|\mathcal{H}^t f\|_2 \leq \mathrm{e}^{-bt}\, \|f\|_2 \qquad \forall f \in L^2(M) \quad \forall t \in \mathbb{R}^+;$$

(c) $\{\mathcal{H}^t\}$ is *ultracontractive*, i.e. for every t in $\mathbb{R}^+$ the operator $\mathcal{H}^t$ maps $L^1(M)$ into $L^\infty(M)$.

Remark 2.2. Assumption (ii) forces $\mu(M) = \infty$. In fact, it forces M to have exponential volume growth (see [**7**, Proposition 2.5 (i)] for details).

Remark 2.3. Assumption (iv) has the following straightforward consequences:

(i) $\{\mathcal{H}^t\}$ is a strongly continuous semigroup of contractions on $L^1(M)+L^2(M)$;

(ii) since for each p in $[1,2]$ the space $L^p(M)$ is continuously embedded in $L^1(M)+L^2(M)$, we may consider the restriction $\mathcal{H}_p^t$ of the operator $\mathcal{H}^t$ to $L^p(M)$. Then $\{\mathcal{H}_p^t\}$ is strongly continuous on $L^p(M)$, and satisfies the estimate

$$\|\mathcal{H}_p^t f\|_p \leq \mathrm{e}^{-2b\,(1-1/p)\,t}\,\|f\|_p \qquad \forall f\in L^p(M) \quad \forall t\in\mathbb{R}^+; \tag{2.2}$$

(iii) by (iv) (a) and (iv) (c) above, for each t in $\mathbb{R}^+$ the operator $\mathcal{H}^t$ maps $L^1(M)$ into $L^1(M)\cap L^2(M)$. Hence $\mathcal{H}^t$ maps $L^1(M)$ into $L^p(M)$ for each p in $[1,2]$.

Denote by $-\mathcal{G}$ the infinitesimal generator of $\{\mathcal{H}^t\}$ on $L^1(M)+L^2(M)$. Since $\{\mathcal{H}^t\}$ is contractive on $L^1(M)+L^2(M)$, the spectrum of $\mathcal{G}$ is contained in the right half plane. Then, for every σ in $\mathbb{R}^+$ we may consider the resolvent operator $(\sigma\mathcal{I}+\mathcal{G})^{-1}$ of $\{\mathcal{H}^t\}$, that we denote by $\mathcal{R}_\sigma$. We denote by $\mathcal{R}_{\sigma,p}$ the restriction of $\mathcal{R}_\sigma$ to $L^p(M)$, and by $-\mathcal{G}_p$ the generator of $\{\mathcal{H}_p^t\}$. Obviously $\mathcal{R}_{\sigma,p}$ is the resolvent of $\{\mathcal{H}_p^t\}$ and $-\mathcal{G}_p$ is the restriction of $-\mathcal{G}$ to $\mathrm{Dom}(\mathcal{G}_p)$, which coincides with $\mathcal{R}_\sigma\big(L^p(M)\big)$.

For every σ in $\mathbb{R}^+$ denote by $\mathcal{U}_\sigma$ the operator $\mathcal{G}\mathcal{R}_\sigma$. Observe that

$$\mathcal{U}_\sigma = \mathcal{I}-\sigma\,\mathcal{R}_\sigma,$$

so that $\mathcal{U}_\sigma$ is bounded on $L^1(M)+L^2(M)$, and its restriction $\mathcal{U}_{\sigma,p}$ to $L^p(M)$ is bounded on $L^p(M)$ for every $p\in[1,2]$. Moreover $\mathcal{U}_\sigma$ and $\mathcal{H}^t$ commute for every t in $\mathbb{R}^+$.

Proposition 2.4. *For each positive integer k the following hold:*

(i) *if p is in $(1,2]$, then the operator $\mathcal{U}_{\sigma,p}^k$ is an isomorphism of $L^p(M)$;*

(ii) *the operator $\mathcal{U}_\sigma^k$ is injective on $L^1(M)+L^2(M)$.*

Proof. First we prove (i). Clearly, it suffices to show that $\mathcal{U}_{\sigma,p}$ is an isomorphism of $L^p(M)$. By (2.2) the bottom of the spectrum of $\mathcal{G}_p$ is positive. Thus $\mathcal{G}_p^{-1}$ and $\sigma\,\mathcal{G}_p^{-1}+\mathcal{I}$ are bounded. Since $\mathcal{U}_{\sigma,p}^{-1}=\mathcal{G}_p^{-1}(\sigma\mathcal{I}+\mathcal{G}_p)$ and $\mathcal{G}_p^{-1}(\sigma\mathcal{I}+\mathcal{G}_p)=\sigma\,\mathcal{G}_p^{-1}+\mathcal{I}$, (i) is proved.

Next we prove (ii). It suffices to prove the result in the case where $k=1$, since the general case follows by induction. Suppose that f is a function in $L^1(M)+L^2(M)$ such that $\mathcal{U}_\sigma f=0$. Then $\mathcal{U}_\sigma\big(\mathcal{H}^t f\big)=\mathcal{H}^t\big(\mathcal{U}_\sigma f\big)=0$ for all t in $\mathbb{R}^+$. By the ultracontractivity of $\mathcal{H}^t$, and the fact that the restriction of $\mathcal{H}^t$ to $L^2(M)$ is bounded on $L^2(M)$, the function $\mathcal{H}^t f$ is in $L^2(M)$ for all t in $\mathbb{R}^+$. Thus $\mathcal{U}_\sigma\big(\mathcal{H}^t f\big)=\mathcal{U}_{\sigma,2}\big(\mathcal{H}^t f\big)=0$. Hence $\mathcal{H}^t f=0$, because $\mathcal{U}_{\sigma,2}$ is an isomorphism. Since $\{\mathcal{H}^t\}$ is strongly continuous on $L^1(M)+L^2(M)$ by Remark 2.3 (i), $\mathcal{H}^t f$ tends to f in $L^1(M)+L^2(M)$ as t tends to 0, and (ii) follows. □

We recall the definitions of the atomic Hardy space $H^1(M)$ and its dual space $BMO(M)$ given in [**7**].

Definition 2.5.
An H^1-*atom* a is a function in $L^1(M)$ supported in a ball B with the following properties:

(i) $\int_B a\,\mathrm{d}\mu=0$;

(ii) $\|a\|_2\leq\mu(B)^{-1/2}$.

Definition 2.6. Suppose that s is in $\mathbb{R}^+$. The *Hardy space* $H^1_s(M)$ is the space of all functions g in $L^1(M)$ that admit a decomposition of the form

$$g = \sum_{k=1}^{\infty} \lambda_k \, a_k, \tag{2.3}$$

where a_k is a H^1-atom *supported in a ball B of $\mathcal{B}_s$*, and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. The norm $\|g\|_{H^1_s}$ of g is the infimum of $\sum_{k=1}^{\infty} |\lambda_k|$ over all decompositions (2.3) of g.

The vector space $H^1_s(M)$ is independent of s in $\big(R_0/(1-\gamma), \infty\big)$, where R_0 and γ are as in Basic assumptions 2.1 (iii) (see [**7**, Proposition 5.1]). Furthermore, given s_1 and s_2 in $\big(R_0/(1-\gamma), \infty\big)$, the norms $\|\cdot\|_{H^1_{s_1}}$ and $\|\cdot\|_{H^1_{s_2}}$ are equivalent.

Notation. We shall denote the space $H^1_s(M)$ simply by $H^1(M)$, and we endow $H^1(M)$ with the norm $H^1_{s_0}(M)$, where $s_0 = \max\big(R_0/(1-\gamma), 1\big)$. We note explicitly that if $R_0 = 0$, then $s_0 = 1$.

The Banach dual of $H^1(M)$ is isomorphic [**7**, Thm 5.1] to the space $BMO(M)$, which we now define.

Definition 2.7. The space $BMO(M)$ is the space of all locally integrable functions f such that $N(f) < \infty$, where

$$N(f) = \sup_{B \in \mathcal{B}_{s_0}} \frac{1}{\mu(B)} \int_B |f - f_B| \, \mathrm{d}\mu,$$

and f_B denotes the average of f over B. We endow $BMO(M)$ with the "norm"

$$\|f\|_{BMO} = N(f).$$

Remark 2.8. It is straightforward to check that f is in $BMO(M)$ if and only if its sharp maximal function $f^\sharp$, defined by

$$f^\sharp(x) = \sup_{B \in \mathcal{B}_{s_0}(x)} \frac{1}{\mu(B)} \int_B |f - f_B| \, \mathrm{d}\mu \qquad \forall x \in M,$$

is in $L^\infty(M)$. Here $\mathcal{B}_{s_0}(x)$ denotes the family of all balls in $\mathcal{B}_{s_0}$ that contain the point x.

In the last part of this section we define the new spaces $X^k_\sigma(M)$ of Hardy type and their dual spaces $Y^k_\sigma(M)$, and prove an interpolation result, which is relevant for later developments.

Definition 2.9. For each positive integer k and for each σ in $\mathbb{R}^+$ we denote by $X^k_\sigma(M)$ the Banach space of all $L^1(M)$ functions f such that $\mathcal{U}_\sigma^{-k} f$ is in $H^1(M)$, endowed with the norm

$$\|f\|_{X^k} = \|\mathcal{U}_\sigma^{-k} f\|_{H^1}.$$

Note that $\mathcal{U}_\sigma^{-k}$ is, by definition, an isometric isomorphism between $X^k_\sigma(M)$ and $H^1(M)$. In Section 3, we shall see that $X^k_\sigma(M)$ may be characterised as the image of $H^1(M)$ under a wide class of maps $\mathcal{V}^k$.

REMARK 2.10. Note that the space $X^k_\sigma(M)$ is continuously included in $L^1(M)$. Indeed, suppose that f is in $X^k_\sigma(M)$. Then

$$\begin{aligned}\|f\|_1 = \big\|\mathcal{U}^k_\sigma \mathcal{U}^{-k}_\sigma f\big\|_1 \leq \big|\!\big|\!\big|\mathcal{U}^k_\sigma\big|\!\big|\!\big|_1 \, \big\|\mathcal{U}^{-k}_\sigma f\big\|_1 &\leq \big|\!\big|\!\big|\mathcal{U}^k_\sigma\big|\!\big|\!\big|_1 \, \big\|\mathcal{U}^{-k}_\sigma f\big\|_{H^1} \\ &= \big|\!\big|\!\big|\mathcal{U}^k_\sigma\big|\!\big|\!\big|_1 \, \big\|f\big\|_{X^k_\sigma},\end{aligned}$$

as required. Note that the last inequality is a consequence of the fact that $H^1(M)$ is continuously included in $L^1(M)$.

DEFINITION 2.11. For each positive integer k, and for each σ in $\mathbb{R}^+$ we denote by $Y^k_\sigma(M)$ the Banach dual of $X^k_\sigma(M)$.

REMARK 2.12. Since $\mathcal{U}^{-k}_\sigma$ is an isometric isomorphism between $X^k_\sigma(M)$ and $H^1(M)$, its adjoint map $\big(\mathcal{U}^{-k}_\sigma\big)^*$ is an isometric isomorphism between $BMO(M)$ and $Y^k_\sigma(M)$. Hence

$$\big\|\big(\mathcal{U}^{-k}_\sigma\big)^* f\big\|_{Y^k_\sigma} = \|f\|_{BMO}.$$

Given a compatible couple of Banach spaces X_0 and X_1 we denote by $(X_0, X_1)_{[\theta]}$ its complex interpolation space, also denoted by X_θ.

PROPOSITION 2.13. *Suppose that (X^0, X^1) and (Y^0, Y^1) are interpolation pairs of Banach spaces. Suppose further that $\mathcal{T}$ is a bounded linear map from $X^0 + X^1$ to $Y^0 + Y^1$, such that the restrictions $\mathcal{T} : X^0 \to Y^0$ and $\mathcal{T} : X^1 \to Y^1$ are isomorphisms. Then for every θ in $(0,1)$ the restriction $\mathcal{T} : X_\theta \to Y_\theta$ is an isomorphism.*

Proof. For every θ in $[0,1]$ denote by $\mathcal{T}_\theta$ the restriction of $\mathcal{T}$ to X_θ. Define $\mathcal{S} : Y_0 + Y_1 \to X_0 + X_1$ by setting

$$\mathcal{S}(y_0 + y_1) = \mathcal{T}_0^{-1} y_0 + \mathcal{T}_1^{-1} y_1.$$

It is straightforward to check that the operator $\mathcal{S}$ is well defined, bounded and linear. Moreover $\mathcal{S}\mathcal{T}$ is the identity on $X_0 + X_1$ and $\mathcal{T}\mathcal{S}$ is the identity on $Y_0 + Y_1$. Thus $\mathcal{S} = \mathcal{T}^{-1}$. Hence $\mathcal{S}_\theta = \mathcal{T}_\theta^{-1}$. Finally, $\mathcal{S}_\theta : Y_\theta \to X_\theta$ is bounded by interpolation. This concludes the proof of the proposition. □

THEOREM 2.14. *Suppose that σ is in $\mathbb{R}^+$, k is a positive integer, and θ is in $(0,1)$. The following hold:*

(i) *if $1/p = 1 - \theta/2$, then $\big(X^k_\sigma(M), L^2(M)\big)_{[\theta]} = L^p(M)$ with equivalent norms;*

(ii) *if $1/q = (1-\theta)/2$, then $\big(L^2(M), Y^k_\sigma(M)\big)_{[\theta]} = L^q(M)$ with equivalent norms.*

Proof. To prove (i), we first observe that $\mathcal{U}^k_\sigma$ is an isomorphism of $H^1(M) + L^2(M)$ onto $X^k_\sigma(M) + L^2(M)$. Then we may apply Proposition 2.13 with $\mathcal{U}^k_\sigma$ in place of $\mathcal{T}$, $X^0 = H^1(M)$,

$Y^0 = X^k_\sigma(M)$, $X^1 = L^2(M) = Y^1$. By [**7**, Thm 7.4]

$$\big(H^1(M), L^2(M)\big)_{[\theta]} = L^p(M).$$

By Proposition 2.13, the restriction of $\mathcal{U}^k_\sigma$ to $L^p(M)$ is an isomorphism between $L^p(M)$ and $\big(X^k_\sigma(M), L^2(M)\big)_{[\theta]}$. But the restriction of $\mathcal{U}^k_\sigma$ to $L^p(M)$ is just $\mathcal{U}^k_{\sigma,p}$, which is an isomorphism of $L^p(M)$ by Proposition 2.4. Hence $\big(X^k_\sigma(M), L^2(M)\big)_{[\theta]}$ and $L^p(M)$ are isomorphic Banach spaces, as required.

Now (ii) follows from (i) by the duality theorem. □

3. *New Hardy spaces on manifolds*

Suppose that M is a connected n-dimensional Riemannian manifold of infinite volume with Riemannian measure μ.

Basic assumptions 3.1. We make the following assumptions on M:
(i) $b > 0$;
(ii) $\mathrm{Ric} \geq -\kappa^2$ for some positive κ and the injectivity radius is positive.

Remark 3.2. It is well known that manifolds with properties (i)-(ii) above satisfy the *uniform ball size condition*, i.e.,

$$\inf\big\{\mu\big(B(p,r)\big) : p \in M\big\} > 0 \qquad \text{and} \qquad \sup\big\{\mu\big(B(p,r)\big) : p \in M\big\} < \infty.$$

See, for instance, [**17**], where complete references are given.

Note that manifolds satisfying the assumptions above also satisfy the Basic assumptions 2.1. Indeed, every length metric space satisfies the *approximate midpoint property* (AM), and, by standard comparison theorems [**9**, Thm 3.10], the measure μ is *locally doubling*. Furthermore, it is known [**7**, Section 8] that for manifolds with Ricci curvature bounded from below the assumption $b > 0$ is equivalent to the *isoperimetric property* (I). Finally, the heat semigroup $\{\mathcal{H}^t\}$ possesses the properties (iv) (a)–(c) of the Basic Assumptions 2.1 [**19**].

In this section we complement the theory developed in Section 2 by proving that the spaces $X^k_\sigma(M)$ and $Y^k_\sigma(M)$, in fact, do not depend on σ as long as $\sigma > \beta^2 - b$ (see Theorem 3.5). Our main tool for proving this is a $H^1(M)$ boundedness result, of independent interest, for functions of the Laplace–Beltrami operator on M (Theorem 3.4), which will also play an important role in the proof of Theorem 4.3.

Recall that $-\mathcal{L}$, b and β denote the Laplace–Beltrami operator on M, the bottom of the $L^2(M)$ spectrum of $\mathcal{L}$, and the exponential rate of growth of the volume of geodesic balls (see (1.1)) respectively. By a result of Brooks [**6**] $b \leq \beta^2$. Further, denote by δ a nonnegative number such that the following ultracontractive estimate [**19**, Section 7.5] holds

$$|||\mathcal{H}^t|||_{1;2} \leq C\,\mathrm{e}^{-bt}\,t^{-n/4}\,(1+t)^{n/4-\delta/2} \qquad \forall t \in \mathbb{R}^+. \tag{3.1}$$

First we define an appropriate function space of holomorphic functions which will be needed in the statement of Theorem 3.4.

Definition 3.3. Suppose that J is a positive integer and that W is in $\mathbb{R}^+$. Denote by $\mathbf{S}_W$ the strip $\{\zeta \in \mathbb{C} : \mathrm{Im}(\zeta) \in (-W, W)\}$ and by $H^\infty(\mathbf{S}_W; J)$ the vector space of all bounded *even* holomorphic functions f in $\mathbf{S}_W$ for which there exists a positive constant C such that

$$|D^j f(\zeta)| \leq C\,(1+|\zeta|)^{-j} \qquad \forall \zeta \in \mathbf{S}_W \quad \forall j \in \{0,1,\ldots,J\}. \tag{3.2}$$

We denote by $\|f\|_{\mathbf{S}_W;J}$ the infimum of all constants C for which (3.2) holds.

NOTATION. For the sake of notational simplicity, we denote by $\mathcal{D}$ the operator $\sqrt{\mathcal{L}-b}$.

THEOREM 3.4. *Assume that α and β are as in (1.1), and δ as in (3.1). Denote by N the integer $[n/2+1]+1$. Suppose that J is an integer $> \max\big(N+2+\alpha/2-\delta, N+1/2\big)$. Then there exists a constant C such that*

$$\|m(\mathcal{D})\|_{H^1} \leq C\,\|m\|_{\mathbf{S}_\beta;J} \qquad \forall m \in H^\infty\big(\mathbf{S}_\beta;J\big).$$

We emphasise that the width of the strip in Theorem 3.4 is best possible as the case of symmetric spaces of the noncompact type shows [**11**]. Note that if M is a symmetric space of the noncompact type with rank r and $\mathcal{H}^t$ denotes the semigroup associated to the Killing metric, then δ is equal to the sum of $r/2$ and the cardinality of the positive indivisible restricted roots [**15**, Thm 3.2 (iii)], and $\alpha = (r-1)/2$. Thus, in this case, we need only to assume $J > N+1/2$ in Theorem 3.4.

Our result may be compared with [**40**, Corollary B.3], where the author proved, under much stronger curvature assumptions on M, that if m is in the symbol class $\mathcal{S}^0_{\beta^2}$, then $m(\mathcal{D})$ maps the Goldberg type space $\mathfrak{h}^1(M)$ to $L^1(M)$ and $L^\infty(M)$ into $\mathfrak{bmo}(M)$.

The proof of Theorem 3.4 is fairly technical and will be given in Section 5. An important consequence of Theorem 3.4 is that, for fixed k, the spaces $X^k_\sigma(M)$ do not depend on the parameter σ, as σ varies in (β^2-b,∞).

THEOREM 3.5. *The following hold:*

(i) *if σ_1 and σ_2 are in (β^2-b,∞), then $X^k_{\sigma_1}(M)$ and $X^k_{\sigma_2}(M)$ agree as vector spaces, and their norms are equivalent;*
(ii) *if σ is in (β^2-b,∞), then $H^1(M) \supset X^1_\sigma(M) \supset X^2_\sigma(M) \supset \cdots$ with continuous inclusions;*
(iii) *the inclusions in* (ii) *are proper.*

Proof. First we prove (i). Consider the operator $\mathcal{T}_{\sigma_1,\sigma_2}$, defined on $L^2(M)$ by

$$\mathcal{T}_{\sigma_1,\sigma_2} = \mathcal{U}_{\sigma_1}^{-1}\,\mathcal{U}_{\sigma_2}.$$

Since both $\mathcal{U}_{\sigma_1}$ and $\mathcal{U}_{\sigma_2}$ are isomorphisms on $L^2(M)$, so are $\mathcal{T}_{\sigma_1,\sigma_2}$ and $\mathcal{T}^{-1}_{\sigma_1,\sigma_2}$. Observe that the operators $\mathcal{T}_{\sigma_1,\sigma_2}$ and $\mathcal{T}^{-1}_{\sigma_1,\sigma_2}$ are bounded on $H^1(M)$. Indeed,

$$\mathcal{T}_{\sigma_1,\sigma_2} = (\sigma_1\,\mathcal{I}+\mathcal{L})\,(\sigma_2\,\mathcal{I}+\mathcal{L})^{-1} = (\sigma_1-\sigma_2)\,(\sigma_2\,\mathcal{I}+\mathcal{L})^{-1}+\mathcal{I}.$$

Hence the boundedness of $\mathcal{T}_{\sigma_1,\sigma_2}$ on $H^1(M)$ is equivalent to that of $(\sigma_2\,\mathcal{I}+\mathcal{L})^{-1}$. To prove that $(\sigma_2\,\mathcal{I}+\mathcal{L})^{-1}$ is bounded on $H^1(M)$, it suffices to check that the associated spectral multiplier $\zeta \mapsto (\sigma+b+\zeta^2)^{-1}$ satisfies the hypotheses of Theorem 3.4. We omit the details of this calculation. A similar argument shows that $\mathcal{T}^{-1}_{\sigma_1,\sigma_2}$ is bounded on $H^1(M)$.

Thus, $\mathcal{T}_{\sigma_1,\sigma_2}$ is an isomorphism of $H^1(M)$. Since $\mathcal{U}_{\sigma_1}\mathcal{T}_{\sigma_1,\sigma_2}\mathcal{U}_{\sigma_2}^{-1} = \mathcal{I}$, the identity is an isomorphism between $X^1_{\sigma_1}(M)$ and $X^1_{\sigma_2}(M)$, as required to conclude the proof of (i) in the case where $k = 1$. The proof in the case where $k \geq 2$ is similar, and is omitted.

Note that (i) is equivalent to the boundedness of $\mathcal{U}_\sigma$ on $H^1(M)$. Since $\mathcal{U}_\sigma = \mathcal{I} - \sigma\,(\sigma\mathcal{I} + \mathcal{L})^{-1}$, it suffices to prove that the resolvent operator $(\sigma\mathcal{I} + \mathcal{L})^{-1}$ is bounded on $H^1(M)$. This has already been done in the proof of (i), and (ii) follows.

Finally we prove (iii). Choose a function ψ in $C^\infty_c(M)$ with nonvanishing integral. Observe that $\mathcal{L}\psi$ is a multiple of a H^1-atom, hence it is in $H^1(M)$.

We shall prove that $\mathcal{L}^{k+1}\psi$ is in $X^k_\sigma(M) \setminus X^{k+1}_\sigma(M)$. Indeed, on the one hand

$$\mathcal{U}_\sigma^{-k}\big(\mathcal{L}^{k+1}\psi\big) = (\sigma\mathcal{I} + \mathcal{L})^k\big(\mathcal{L}\psi\big),$$

which again is a multiple of an H^1-atom, hence is in $H^1(M)$. On the other hand

$$\mathcal{U}_\sigma^{-(k+1)}\big(\mathcal{L}^{k+1}\psi\big) = (\sigma\mathcal{I} + \mathcal{L})^{k+1}(\psi),$$

which may be written as a linear combination of ψ and of terms of the form $\mathcal{L}^j\psi$ with j in $\{1, \ldots, k+1\}$. Therefore the integral of $\mathcal{U}_\sigma^{-(k+1)}\big(\mathcal{L}^{k+1}\psi\big)$ does not vanish, hence it is not in $H^1(M)$ and $\mathcal{L}^{k+1}\psi$ is not in $X^{k+1}_\sigma(M)$, as required. □

Definition 3.6. Suppose that k is a positive integer. The space $X^k_{\beta^2}(M)$ will be denoted simply by $X^k(M)$.

By Theorem 3.5, for any σ in $(\beta^2 - b, \infty)$ and each positive integer k we have that $X^k(M) = X^k_\sigma(M)$ as vector spaces, and their norms are equivalent.

Remark 3.7. The space $X^k(M)$ may be characterised as the image of $H^1(M)$ under a wider class of maps. This is done in [**33**, Subection 4.6]. We briefly describe the result.

For each positive ε there exists a function η in $C_c(\mathbb{R})$ such that the only zeroes of $1 - \widehat{\eta}$ in $\overline{\mathbf{S}}_{\beta+\varepsilon}$ are the points $\pm i\sqrt{b}$ (here $\widehat{\eta}$ denotes the Fourier transform of η). Suppose that k is a positive integer. Denote by $\mathcal{V}_\eta$ the operator $\mathcal{I} - \widehat{\eta}(\mathcal{D})$. The following hold:

(i) the map $\mathcal{V}_\eta^k$ is injective on $L^1(M)$;

(ii) $\mathcal{V}_\eta^k H^1(M) = X^k(M)$ as vector spaces, and the norm on $X^k(M)$, defined by

$$\|f\|_{\eta,k} = \|\mathcal{V}_\eta^{-k} f\|_{H^1} \qquad \forall f \in X^k(M),$$

is equivalent to the norm of $X^k(M)$.

4. *Main results*

In this section we state and prove boundedness results for strongly singular spectral multipliers and first order Riesz transform associated to the Laplace–Beltrami operator on complete connected Riemannian manifolds M satisfying the Basic assumptions 3.1.

We recall that in Definition 3.3 we introduced the space $H^\infty(\mathbf{S}_W; J)$ of functions that are holomorphic and bounded, together with their derivatives up to the order J, in the strip $\mathbf{S}_W$, and satisfy a Mihlin-type condition at infinity. Here, to deal with a wider class of operators, we define a larger space of functions that may be singular also at the points $\pm iW$.

DEFINITION 4.1. Suppose that J is a positive integer, that τ is in $[0,\infty)$, and that W is in $\mathbb{R}^+$. The space $H(\mathbf{S}_W; J,\tau)$ is the vector space of all holomorphic *even* functions f in the strip $\mathbf{S}_W$ for which there exists a positive constant C such that

$$|D^j f(\zeta)| \leq C \max\big(|\zeta^2+W^2|^{-\tau-j}, |\zeta|^{-j}\big) \qquad \forall \zeta \in \mathbf{S}_W \quad \forall j \in \{0,1,\ldots,J\}. \tag{4.1}$$

We denote by $\|f\|_{\mathbf{S}_W;J,\tau}$ the infimum of all constants C for which (4.1) holds.

Note that, for each fixed j, the right-hand side of (4.1) is infinite of order $-\tau-j$ at $\pm iW$, and vanishes of order j at infinity. Thus, if $\tau=0$, and f is in $H(\mathbf{S}_W; J,\tau)$, then f satisfies Mihlin-type conditions both near the points $\pm iW$ and at infinity. In particular, the derivatives of f may be unbounded in any neighbourhood of iW, and of $-iW$. Finally, if τ is in $\mathbb{R}^+$, and f is in $H(\mathbf{S}_W; J,\tau)$, then both f and its derivatives up to the order J may be unbounded in any neighbourhood of iW, and of $-iW$.

REMARK 4.2. An interesting example of a function in $H(\mathbf{S}_\beta; J,\tau)$ is

$$m(\zeta) = (\zeta^2+\beta^2)^{-iu-\tau}\,(\zeta^2+\beta^2+1)^{\tau},$$

where τ is in $[0,\infty)$. Note that if $b=\beta^2$, then $m(\mathcal{D}) = \mathcal{L}^{-iu-\tau}\,(\mathcal{L}+\mathcal{I})^{\tau}$. It is worth observing that there are no endpoint results at $p=1$ for this operator in the literature when $\tau>1$. In the case where M is a symmetric space of the noncompact type, it is known [**1**, **3**, **35**] that $m(\mathcal{D})$ is of weak type 1 if and only if $\tau \leq 1$, but the proof of this fact uses the spherical Fourier transform and very specific information on the structure of the symmetric space, and it is hardly extendable.

THEOREM 4.3. *Assume that α and β are as in (1.1), and δ as in (3.1). Suppose that τ is in $[0,\infty)$, that J and k are integers, with $k>\tau+J$ and $J>\max\big(N+2+\alpha/2-\delta, N+1/2\big)$, where N denotes the integer $[\![n/2+1]\!]+1$. The following hold:*

(i) *if $b<\beta^2$, then there exists a constant C such that*

$$|\!|\!| m(\mathcal{D}) |\!|\!|_{H^1;L^1} \leq C\,\|m\|_{\mathbf{S}_\beta;J,\tau} \qquad \forall m \in H(\mathbf{S}_\beta; J,\tau)$$

and

$$|\!|\!| m(\mathcal{D})^t |\!|\!|_{L^\infty;BMO} \leq C\,\|m\|_{\mathbf{S}_\beta;J,\tau} \qquad \forall m \in H(\mathbf{S}_\beta; J,\tau),$$

where $m(\mathcal{D})^t$ denotes the transpose operator of $m(\mathcal{D})$;

(ii) *if $b=\beta^2$, then there exists a constant C such that*

$$|\!|\!| m(\mathcal{D}) |\!|\!|_{X^k;H^1} \leq C\,\|m\|_{\mathbf{S}_\beta;J,\tau} \qquad \forall m \in H(\mathbf{S}_\beta; J,\tau)$$

and

$$|\!|\!| m(\mathcal{D})^t |\!|\!|_{BMO;Y^k} \leq C\,\|m\|_{\mathbf{S}_\beta;J,\tau} \qquad \forall m \in H(\mathbf{S}_\beta; J,\tau),$$

where $m(\mathcal{D})^t$ denotes the transpose operator of $m(\mathcal{D})$.

Proof. First we prove (i). Consider the map $\widetilde{\mathcal{U}}$, defined by

$$\widetilde{\mathcal{U}} = \big[\mathcal{L}+(\beta^2-b)\mathcal{I}\big]\,(\beta^2\mathcal{I}+\mathcal{L})^{-1}.$$

Observe that $\widetilde{\mathcal{U}} = \mathcal{I} - b\,(\beta^2\mathcal{I}+\mathcal{L})^{-1}$ extends to a bounded operator on $L^1(M)$, because the $L^1(M)$-spectrum of $\mathcal{L}$ is contained in the right half-plane. Similarly, the operator $\mathcal{I}+b\,[(\beta^2-$

$b)\mathcal{I} + \mathcal{L}]^{-1}$ extends to a bounded operator on $L^1(M)$; it is straightforward to check that this operator is the inverse of $\widetilde{\mathcal{U}}$ on $L^1(M)$. Thus, $\widetilde{\mathcal{U}}$ is an isomorphism of $L^1(M)$, and so is $\widetilde{\mathcal{U}}^k$.

Consequently, $m(\mathcal{D})$ is bounded from $H^1(M)$ to $L^1(M)$ if and only if $\widetilde{\mathcal{U}}^k m(\mathcal{D})$ is bounded from $H^1(M)$ to $L^1(M)$. Observe that $\widetilde{\mathcal{U}}^k m(\mathcal{D}) = u_k(\mathcal{D})$, where

$$u_k(\zeta) = \Big(\frac{\zeta^2+\beta^2}{\zeta^2+b+\beta^2}\Big)^k\, m(\zeta).$$

It is straightforward to check that there exists a constant C such that

$$|D^j u_k(\zeta)| \leq C\,\|m\|_{\mathbf{S}_\beta;J,\tau}\,\big(1+|\zeta|\big)^{-j} \qquad \forall \zeta\in\mathbf{S}_\beta \quad \forall j\in\{0,1,\ldots,J\}.$$

Here we use the fact that $k > \tau + J$. Thus, $u_k(\mathcal{D})$ is bounded on $H^1(M)$ by Theorem 3.4, hence from $H^1(M)$ to $L^1(M)$, as required to prove the first estimate.

The second follows from the first by a duality argument.

Next we prove (ii). Observe that $m(\mathcal{D}) = m(\mathcal{D})\,\mathcal{U}^k_{\beta^2}\,\mathcal{U}^{-k}_{\beta^2}$. Since $\mathcal{U}^{-k}_{\beta^2}$ is an isometric isomorphism between $X^k(M)$ and $H^1(M)$, to prove that $m(\mathcal{D})$ is bounded from $X^k(M)$ to $H^1(M)$ it suffices to show that the operator $m(\mathcal{D})\,\mathcal{U}^k_{\beta^2}$ extends to a bounded operator on $H^1(M)$. Note that $m(\mathcal{D})\,\mathcal{U}^k_{\beta^2} = v_k(\mathcal{D})$, where

$$v_k(\zeta) = \Big(\frac{\zeta^2+b}{\zeta^2+b+\beta^2}\Big)^k\, m(\zeta).$$

It is straightforward to check that there exists a constant C such that

$$|D^j v_k(\zeta)| \leq C\,\|m\|_{\mathbf{S}_\beta;J,\tau}\,\big(1+|\zeta|\big)^{-j} \qquad \forall \zeta\in\mathbf{S}_\beta \quad \forall j\in\{0,1,\ldots,J\}.$$

Here we use the fact that $k > \tau + J$. Thus, $v_k(\mathcal{D})$ is bounded on $H^1(M)$ by Theorem 3.4, as required to prove the first estimate. The second follows from the first by a duality argument.

The proof of the theorem is complete. □

REMARK 4.4. Assume that M has C^∞ bounded geometry. By proceeding as in the proof of Theorem 4.3 and using [**7**, Thm 10.2] instead Theorem 3.4, we may prove Theorem 4.3 (i) with $J > \max(\alpha+1, n/2+1)$ in place of $J > \max\big(N+2+\alpha/2-\delta, N+1/2\big)$.

COROLLARY 4.5. *Suppose that M is a symmetric space of the noncompact type and that $-\mathcal{L}$ is the Laplace–Beltrami operator with respect to the Killing metric. If $k > n/2+3$, then $\mathcal{L}^{iu}$ is bounded from $X^k(M)$ to $H^1(M)$.*

Proof. Indeed, it is well known that $\alpha = (r-1)/2$, where r is the rank of the symmetric space, and $\delta = \upsilon + r/2$, where υ denotes the cardinality of the indivisible positive restricted roots. Notice that $3/2+\alpha/2-\delta \leq 0$, so that the hypotheses of Theorem 4.3 are satisfied whenever $J > n/2+2$ and $k > J$, and the required conclusion follows. □

We conclude this section with the following endpoint result for the first order Riesz transform. Our method hinges on the fact that if $b = \beta^2$ and k is large enough, then the operator $\mathcal{L}^k\,(\beta^2\mathcal{I} + \mathcal{L})^{-k}$ is bounded on $H^1(M)$ by Theorem 3.4.

Theorem 4.6. *Assume that α and β are as in (1.1), and δ as in (3.1). Suppose that $b = \beta^2$ and that k is an integer $> \max\big(N+2+\alpha/2-\delta, N+1/2\big)$, where N denotes the integer $[\![n/2+1]\!]+1$. Then the first order Riesz transform $\nabla\mathcal{L}^{-1/2}$ is bounded from $X^k(M)$ to $L^1(M)$.*

Proof. Since $\mathcal{L}^k\,(\beta^2\mathcal{I}+\mathcal{L})^{-k}$ is an isometry between $H^1(M)$ and $X^k(M)$, it suffices to prove that $\nabla\mathcal{L}^{k-1/2}\,(\beta^2\mathcal{I}+\mathcal{L})^{-k}$ is bounded from $H^1(M)$ to $L^1(M)$. Observe that

$$\nabla\mathcal{L}^{k-1/2}\,(\beta^2\mathcal{I}+\mathcal{L})^{-k} = \nabla(\beta^2\mathcal{I}+\mathcal{L})^{-1/2}\,\mathcal{L}^{k-1/2}\,(\beta^2\mathcal{I}+\mathcal{L})^{1/2-k}.$$

The right hand side is the composition of the operators $\mathcal{L}^{k-1/2}\,(\beta^2\mathcal{I}+\mathcal{L})^{1/2-k}$, which is bounded on $H^1(M)$ by Theorem 3.4 and of the translated Riesz transform $\nabla(\beta^2\mathcal{I}+\mathcal{L})^{-1/2}$, which is bounded from $H^1(M)$ to $L^1(M)$ by [**36**]. The required result follows. □

5. *Operators bounded on $H^1(M)$*

This section is devoted to the proof of Theorem 3.5 and is divided in the following subsections: Subsection 5.1, which contains few preliminary results in one dimensional Fourier analysis; Subsection 5.2, where we explain the rôle of the wave propagator in the decomposition into atoms of the image $\mathcal{T}a$ of an H^1-atom a by an operator $\mathcal{T}$; Subsection 5.3, where we prove an economical decomposition of H^1-atoms with "big" support into H^1-atoms with support in balls in $\mathcal{B}_1$; Subsection 5.4, where we prove Theorem 3.4.

5.1. *Some lemmata*

This subsection contains a few technical lemmata concerning one-dimensional Fourier analysis. Some related material may be found in [**30**, Subsection 2.3], which we shall sometimes refer to, for a discussion of the motivations behind this rather technical development.

For every f in $L^1(\mathbb{R})$ define its Fourier transform $\widehat{f}$ by

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(s)\,e^{-ist}\,\mathrm{d}s \qquad \forall t\in\mathbb{R}.$$

Suppose that f is a function on $\mathbb{R}$, and that λ is in $\mathbb{R}^+$. We denote by f^λ and f_λ the λ-dilates of f, defined by

$$f^\lambda(x) = f(\lambda x) \qquad \text{and} \qquad f_\lambda(x) = \lambda^{-1}\,f(x/\lambda) \qquad \forall x\in\mathbb{R}. \tag{5.1}$$

For each $\nu \geq -1/2$, denote by $\mathcal{J}_\nu : \mathbb{R}\setminus\{0\}\to\mathbb{C}$ the modified Bessel function of order ν, defined by

$$\mathcal{J}_\nu(t) = \frac{J_\nu(t)}{t^\nu},$$

where J_ν denotes the standard Bessel function of the first kind and order ν (see, for instance, [**28**, formula (5.10.2), p. 114] for the definition). Recall that

$$\mathcal{J}_{-1/2}(t) = \sqrt{\frac{2}{\pi}}\,\cos t \qquad \text{and that} \qquad \mathcal{J}_{1/2}(t) = \sqrt{\frac{2}{\pi}}\,\frac{\sin t}{t}.$$

For each positive integer ℓ, we denote by $\mathcal{O}^\ell$ the differential operator $t^\ell\,D^\ell$ on the real line.

Lemma 5.1. *For every positive integer k there exists a polynomial P_{k+1} of degree $k+1$ without constant term, such that*

$$\int_{-\infty}^{\infty} f(t)\,\cos(vt)\,\mathrm{d}t = \int_{-\infty}^{\infty} P_{k+1}(\mathcal{O})f(t)\,\mathcal{J}_{k+1/2}(tv)\,\mathrm{d}t, \tag{5.2}$$

for all functions f such that $\mathcal{O}^\ell f \in L^1(\mathbb{R})\cap C_0(\mathbb{R})$ for all ℓ in $\{0,1,\ldots,k+1\}$.

Proof. The proof uses the definition and some properties of the generalised Riesz means $R_{d,z}$, introduced in [**16**, Section 1]. We refer the reader to [**30**, Section 2] for all the prerequisites needed here. In particular, recall that $R_{3+2k,0} = R_{3+2k,-k}R_{3,k}$ by [**30**, Lemma 2.3 (i)]. Now, by integrating by parts and using [**30**, Lemma 2.3 (i) and (ii)],

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,\cos(vt)\,\mathrm{d}t &= -\int_{-\infty}^{\infty} \mathcal{O}f(t)\,\frac{\sin(vt)}{vt}\,\mathrm{d}t\\
&= -\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} \mathcal{O}f(t)\,\big(R_{3+2k,0}\mathcal{J}^v_{1/2}\big)(t)\,\mathrm{d}t\\
&= -\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} R^*_{3+2k,-k}\big(\mathcal{O}f\big)(t)\,\big(R_{3,k}\mathcal{J}^v_{1/2}\big)(t)\,\mathrm{d}t\end{aligned}$$

for all v in $\mathbb{R}$. Furthermore, the definitions of $R_{3,k}$ and of $\mathcal{J}_{1/2}$ and an integration by parts show that

$$\begin{aligned}\big(R_{3,k}\mathcal{J}_{1/2}\big)(u) &= \frac{2}{\Gamma(k)}\,\frac{1}{u}\int_0^1 s\,(1-s^2)^{k-1}\sqrt{\frac{2}{\pi}}\,\sin(su)\,\mathrm{d}s\\
&= \sqrt{\frac{2}{\pi}}\,\frac{1}{\Gamma(k+1)}\int_0^1 (1-s^2)^k\,\cos(su)\,\mathrm{d}s\\
&= 2^k\,\mathcal{J}_{k+1/2}(u).\end{aligned}$$

By [**30**, Lemma 2.4 (i)] there exist constants c_ℓ such that $R^*_{3+2k,-k}\big(\mathcal{O}f\big)= \sum_{\ell=0}^k c_\ell\,\mathcal{O}^{\ell+1}f$, so that

$$\int_{-\infty}^{\infty} f(t)\,\cos(vt)\,\mathrm{d}t = \sum_{\ell=0}^{k} c'_\ell \int_{-\infty}^{\infty} \mathcal{O}^\ell\big(\mathcal{O}f\big)(t)\,\mathcal{J}_{k+1/2}(tv)\,\mathrm{d}t,$$

and the required formula, with $P_{k+1}(s) = \sum_{\ell=0}^k c'_\ell\, s^{\ell+1}$, follows. □

Remark 5.2. We shall denote by $P_{k+1}(\mathcal{O})^*$ the formal adjoint of the operator $P_{k+1}(\mathcal{O})$, i.e. the operator defined by

$$\int_{-\infty}^{\infty} f(t)\,P_{k+1}(\mathcal{O})^*g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} P_{k+1}(\mathcal{O})f(t)\,g(t)\,\mathrm{d}t \qquad \forall f,g\in C_c^\infty(\mathbb{R}).$$

Note that $P_{k+1}(\mathcal{O})^*$ is still a polynomial of degree $k+1$ in $\mathcal{O}$ and that $P_{k+1}(\mathcal{O})^*\mathcal{J}_{k+1/2}(vt) = \cos(vt)$, by (5.2).

Denote by ω an even function in $C_c^\infty(\mathbb{R})$ which is supported in $[-3/4,3/4]$, is equal to 1 in $[-1/4,1/4]$, and satisfies

$$\sum_{j\in\mathbb{Z}} \omega(t-j) = 1 \qquad \forall t\in\mathbb{R}.$$

Denote by ϕ the function $\omega^{1/4} - \omega$, where $\omega^{1/4}$ denotes the 1/4-dilate of ω. Then ϕ is smooth, even and vanishes in the complement of the set $\{t \in \mathbb{R} : 1/4 \le |t| \le 4\}$. For a fixed R in $(0,1]$ and for each positive integer i, denote by E_i the set $\{t \in \mathbb{R} : 4^{i-1}R \le |t| \le 4^{i+1}R\}$. Clearly $\phi^{1/(4^i R)}$ is supported in E_i, and $\sum_{i=1}^{\infty} \phi^{1/(4^i R)} = 1$ in $\mathbb{R} \setminus (-R, R)$. Denote by d the integer $[\![\log_4(3/R)]\!] + 1$. To avoid cumbersome notation, we write ρ_i instead of $1/(4^i R)$. Then

$$\omega^{\rho_0} + \sum_{i=1}^{d} \phi^{\rho_i} = 1 \qquad \text{on} \quad [-3,3]. \tag{5.3}$$

Definition 5.3. We say that a function $g : \mathbb{R} \to \mathbb{C}$ satisfies a *Mihlin condition* [**24**] of order J at infinity on the real line if there exists a constant C such that

$$|D^\ell g(t)| \le C\,(1+|t|)^{-\ell} \qquad \forall t \in \mathbb{R} \quad \forall \ell \in \{0, \dots, J\}. \tag{5.4}$$

We denote by $\|g\|_{\mathrm{Mih}(J)}$ the infimum of all constants C for which (5.4) holds.

Lemma 5.4. *Suppose that k is a nonnegative integer, and that K is an even tempered distribution on $\mathbb{R}$ such that $\|\widehat{K}\|_{\mathrm{Mih}(k+2)}$ is finite. The following hold:*

(i) *for each ℓ in $\{0, \dots, k\}$ the function $t\,\mathcal{O}^\ell K$ is in $L^\infty(\mathbb{R})$, and there exists a constant C such that*

$$\|t\,\mathcal{O}^\ell K\|_\infty \le C\,\|\widehat{K}\|_{\mathrm{Mih}(k+2)} \qquad \forall \ell \in \{0, \dots, k\};$$

(ii) *if $k \ge 1$ and the support of K is contained in $[-1,1]$, then $\widehat{K} = \sum_{i=0}^{d} S_i$, where the functions $S_i : \mathbb{R} \to \mathbb{C}$ are defined by*

$$S_0(\lambda) = (\widehat{\omega}_{\rho_0} * \widehat{K})(\lambda) + \sum_{j=1}^{k} c_{j,k} \int_{-\infty}^{\infty} K(t)\,\mathcal{O}^j \omega(\rho_0 t)\,\mathcal{O}^{k-j} \mathcal{J}_{k+1/2}(\lambda t)\,\mathrm{d}t \tag{5.5}$$

for suitable constants $c_{j,k}$, and, for i in $\{1, \dots, d\}$,

$$S_i(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi^{\rho_i}(t)\,P_{k+1}(\mathcal{O})K(t)\,\mathcal{J}_{k+1/2}(\lambda t)\,\mathrm{d}t; \tag{5.6}$$

(iii) *if the support of K is contained in $[-1,1]$, then there exists a constant C such that*

$$\|S_0\|_\infty \le C\,\|\widehat{K}\|_{\mathrm{Mih}(2)}.$$

Proof. First we prove (i) in the case where $k = 0$. Since $\widehat{K}$ satisfies a Mihlin condition of order 2 at infinity, $D^2\widehat{K}$ is in $L^1(\mathbb{R})$ (see (5.4)), and we may define $F : \mathbb{R} \to \mathbb{C}$ by

$$F(t) = \int_{-\infty}^{\infty} D^2\widehat{K}(\zeta)\,\mathrm{e}^{i\zeta t}\,\mathrm{d}\zeta.$$

By elementary Fourier analysis $tK(t) = -t^{-1}\,F(t)$. Observe that $F(0) = 0$, because

$$\begin{aligned} F(0) &= \lim_{A\to\infty} \int_{-A}^{A} D^2\widehat{K}(\zeta)\,\mathrm{d}\zeta \\ &= 2 \lim_{A\to\infty} D\widehat{K}(A) \\ &= 0, \end{aligned}$$

where we have used the fact that K is even and $D\widehat{K}$ vanishes at infinity, because $\|\widehat{K}\|_{\mathrm{Mih}(2)}$ is finite. Furthermore

$$\begin{aligned} F(t) &= F(t) - F(0) \\ &= \int_{-\infty}^{\infty} D^2\widehat{K}(\zeta)\,(\mathrm{e}^{i\zeta t} - 1)\,\mathrm{d}\zeta. \end{aligned}$$

Suppose that t is positive. Then we write the last integral as the sum of the integrals over the sets $\{\zeta \in \mathbb{R} : |\zeta| \leq 1/t\}$ and $\{\zeta \in \mathbb{R} : |\zeta| > 1/t\}$, and estimate them separately.

To treat the first we integrate by parts, and obtain

$$\begin{aligned} &\int_{|\zeta|\leq 1/t} D^2\widehat{K}(\zeta)\,(\mathrm{e}^{i\zeta t} - 1)\,\mathrm{d}\zeta \\ &= D\widehat{K}(1/t)\,(\mathrm{e}^{i} - 1) - D\widehat{K}(-1/t)\,(\mathrm{e}^{-i} - 1) - it\int_{|\zeta|\leq 1/t} D\widehat{K}(\zeta)\,\mathrm{e}^{i\zeta t}\,\mathrm{d}\zeta. \end{aligned}$$

Since $D\widehat{K}$ is odd, its integral over $[-1/t, 1/t]$ vanishes, so that the last integral may be rewritten as

$$\int_{|\zeta|\leq 1/t} D\widehat{K}(\zeta)\,(\mathrm{e}^{i\zeta t} - 1)\,\mathrm{d}\zeta.$$

Hence

$$\begin{aligned} &\Big|\int_{|\zeta|\leq 1/t} D^2\widehat{K}(\zeta)\,(\mathrm{e}^{i\zeta t} - 1)\,\mathrm{d}\zeta\Big| \\ &\leq C\,\|\widehat{K}\|_{\mathrm{Mih}(2)}\,\frac{|t|}{1+|t|} + C\,t^2\int_{|\zeta|\leq 1/t} |\zeta\, D\widehat{K}(\zeta)|\,\mathrm{d}\zeta \\ &\leq C\,\|\widehat{K}\|_{\mathrm{Mih}(2)}\,|t| \qquad \forall t \in \mathbb{R}^+. \end{aligned}$$

To estimate the second, write

$$\begin{aligned} \Big|\int_{|\zeta|> 1/t} D^2\widehat{K}(\zeta)\,(\mathrm{e}^{i\zeta t} - 1)\,\mathrm{d}\zeta\Big| &\leq C\,\|\widehat{K}\|_{\mathrm{Mih}(2)}\int_{|\zeta|>1/t} \frac{1}{1+\zeta^2}\,\mathrm{d}\zeta \\ &\leq C\,\|\widehat{K}\|_{\mathrm{Mih}(2)}\,|t| \qquad \forall t \in \mathbb{R}^+. \end{aligned}$$

Finally, since K is even,

$$\begin{aligned} \|tK\|_\infty &\leq \sup_{t\in\mathbb{R}} \frac{|F(t)|}{|t|} \\ &\leq C\,\|\widehat{K}\|_{\mathrm{Mih}(2)}, \end{aligned}$$

as required to conclude the proof of (i) in the case where $k = 0$.

Next we assume that $k \geq 1$. By the case $k = 0$ applied to $\mathcal{O}^\ell K$, we see that

$$\|t\,\mathcal{O}^\ell K\|_\infty \leq C\,\|\widehat{\mathcal{O}^\ell K}\|_{\mathrm{Mih}(2)}.$$

Since $\widehat{\mathcal{O}^\ell K} = \sum_{j=0}^{\ell} \alpha_{j,\ell}\,\mathcal{O}^j\widehat{K}$ for suitable constants $\alpha_{j,\ell}$,

$$\begin{aligned} \|\widehat{\mathcal{O}^\ell K}\|_{\mathrm{Mih}(2)} &\leq C\sum_{j=0}^{\ell} \|\mathcal{O}^j\widehat{K}\|_{\mathrm{Mih}(2)} \\ &\leq C\,\|\widehat{K}\|_{\mathrm{Mih}(2+\ell)}, \end{aligned}$$

which is clearly dominated by $C\,\|\widehat{K}\|_{\mathrm{Mih}(k+2)}$, as required to conclude the proof of (i).

Now we prove (ii). Suppose that ε is in $(0,1)$. Clearly $\widehat{K}(\lambda)$ is the limit of $(\widehat{\omega}^{\varepsilon}\,\widehat{K})(\lambda)$ as ε tends to 0. By Fourier inversion formula and Lemma 5.1

$$\begin{aligned}(\widehat{\omega}^{\varepsilon}\,\widehat{K})(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega_{\varepsilon}*K(t)\,\cos(\lambda t)\,\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}P_{k+1}(\mathcal{O})(\omega_{\varepsilon}*K)(t)\,\mathcal{J}_{k+1/2}(\lambda t)\,\mathrm{d}t \qquad \forall\lambda\in\mathbb{R}.\end{aligned}$$

We write the right-hand side as $\sum_{i=0}^{d}S_i(\lambda;\varepsilon)$, where

$$S_0(\lambda;\varepsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^{\rho_0}(t)\,P_{k+1}(\mathcal{O})(\omega_{\varepsilon}*K)(t)\,\mathcal{J}_{k+1/2}(\lambda t)\,\mathrm{d}t \qquad \forall\lambda\in\mathbb{R}, \tag{5.7}$$

and, for each i in $\{1,\ldots,d\}$,

$$S_i(\lambda;\varepsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi^{\rho_i}(t)\,P_{k+1}(\mathcal{O})(\omega_{\varepsilon}*K)(t)\,\mathcal{J}_{k+1/2}(\lambda t)\,\mathrm{d}t \qquad \forall\lambda\in\mathbb{R}.$$

Observe that

$$S_0(\lambda;\varepsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\omega_{\varepsilon}*K)(t)\;P_{k+1}(\mathcal{O})^*(\omega^{\rho_0}\,\mathcal{J}_{k+1/2}^{\lambda})(t)\,\mathrm{d}t.$$

Note that $P_{k+1}(\mathcal{O})^*(\omega^{\rho_0}\,\mathcal{J}_{k+1/2}^{\lambda})$ may be written as

$$\omega^{\rho_0}\,P_{k+1}(\mathcal{O})^*(\mathcal{J}_{k+1/2}^{\lambda}) + \sum_{j=1}^{k}c'_{j,k}\,(\mathcal{O}^j\omega)^{\rho_0}\,(\mathcal{O}^{k-j}\mathcal{J}_{k+1/2})^{\lambda},$$

for suitable constants $c'_{j,k}$, and that $P_{k+1}(\mathcal{O})^*(\mathcal{J}_{k+1/2}^{\lambda})(t) = \cos(t\lambda)$, by Remark 5.2. Hence

$$\begin{aligned}&S_0(\lambda;\varepsilon)\\ &= \big[\widehat{\omega}_{\rho_0}*(\widehat{\omega}^{\varepsilon}\,\widehat{K})\big](\lambda) + \sum_{j=1}^{k}c_{j,k}\int_{-\infty}^{\infty}(\omega_{\varepsilon}*K)(t)\,(\mathcal{O}^j\omega)^{\rho_0}(t)\,(\mathcal{O}^{k-j}\mathcal{J}_{k+1/2})^{\lambda}(t)\,\mathrm{d}t.\end{aligned}$$

Note that for each positive integer j the function $\mathcal{O}^j\omega$ vanishes in $[-1/4,1/4]$, and that the restriction of K to $[-1/4,1/4]^c$ is a bounded function by (i) (with $k=0$). Then it is straightforward to check that $S_0(\lambda;\varepsilon)$ tends to $S_0(\lambda)$ for all λ in $\mathbb{R}$.

To prove that $S_i(\lambda;\varepsilon)$ tends to $S_i(\lambda)$ for all λ in $\mathbb{R}$ and all i in $\{1,\ldots,d\}$, observe that

$$\begin{aligned}2\pi\,S_i(\lambda;\varepsilon) &= \Big\langle\phi^{\rho_i}\,\mathcal{J}_{k+1/2}^{\lambda},P_{k+1}(\mathcal{O})(\omega_{\varepsilon}*K)\Big\rangle\\ &= \Big\langle P_{k+1}(\mathcal{O})^*(\phi^{\rho_i}\,\mathcal{J}_{k+1/2}^{\lambda}),\omega_{\varepsilon}*K\Big\rangle,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between test functions and distributions on $\mathbb{R}$. Now we let ε tend to 0 and obtain

$$\begin{aligned}2\pi\,S_i(\lambda;\varepsilon) &\to \Big\langle P_{k+1}(\mathcal{O})^*(\phi^{\rho_i}\,\mathcal{J}_{k+1/2}^{\lambda}),K\Big\rangle\\ &= \Big\langle\phi^{\rho_i}\,\mathcal{J}_{k+1/2}^{\lambda},P_{k+1}(\mathcal{O})K\Big\rangle.\end{aligned}$$

By (i) the distribution $P_{k+1}(\mathcal{O})K$ is a bounded function on the support of ϕ^{ρ_i}, so that the right hand side is exactly $2\pi\,S_i(\lambda)$, thereby concluding the proof of (ii).

Finally, to prove (iii), observe that

$$
\begin{aligned}
|S_0(\lambda)| &\leq |(\widehat{\omega}_{\rho_0} * \widehat{K})(\lambda)| + C \sum_{j=1}^{k} \int_{-\infty}^{\infty} |K(t)|\, |(\mathcal{O}^j\omega)^{\rho_0}(t)| \,\mathrm{d}t \\
&\leq C\, \|\widehat{K}\|_\infty + C\, \|tK\|_\infty \sum_{j=1}^{k} \int_{-\infty}^{\infty} |t|^{-1}\, |(\mathcal{O}^j\omega)^{\rho_0}(t)| \,\mathrm{d}t \\
&\leq C\, \|\widehat{K}\|_{\mathrm{Mih}(2)} \qquad \forall \lambda \in \mathbb{R},
\end{aligned}
$$

as required. We have used (i) (with $k = 0$) in the second inequality above. □

5.2. *A remark on the wave propagator*

We shall need to prove that certain operators map H^1-atoms into $H^1(M)$. In particular, we need to show that the image of an atom a has integral 0.

Notation. For notational convenience, we denote by $\mathcal{D}_1$ the operator $\sqrt{\mathcal{L} - b + \kappa^2}$ (κ is defined in the Basic assumptions 3.1).

Suppose that $\mathcal{T}$ is an operator bounded on $L^2(M)$. We denote by $k_\mathcal{T}$ its Schwartz kernel (with respect to the Riemannian density μ).

PROPOSITION 5.5. *Suppose that ν is in $[-1/2, \infty)$, that w is in $L^1(\mathbb{R})$, and that a is a H^1-atom. Define the operator $\mathcal{W}_\nu(\mathcal{D})$ on $L^2(M)$ spectrally by*

$$
\mathcal{W}_\nu(\mathcal{D}) f = \int_{-\infty}^{\infty} w(t)\, \mathcal{J}_\nu(t\mathcal{D}) f \,\mathrm{d}t \qquad \forall f \in L^2(M).
$$

The following hold:

(i) $\int_M \mathcal{W}_\nu(\mathcal{D}) a \,\mathrm{d}\mu = 0$;

(ii) $\int_M S_0(\mathcal{D}) a \,\mathrm{d}\mu = 0$ *(S_0 is defined in (5.5)).*

The same conclusions hold if we replace the operator $\mathcal{D}$ by the operator $\mathcal{D}_1$.

Proof. We observe preliminarily that if a is a H^1-atom, then

$$
\int_M \cos(t\mathcal{D}) a \,\mathrm{d}\mu = 0 \qquad \forall t \in \mathbb{R}^+. \tag{5.8}
$$

Indeed, $\cos(t\mathcal{D})a$ is in $L^2(M)$, because $\cos(t\mathcal{D})$ is bounded on $L^2(M)$, and is supported in a ball of radius $t + r_B$, where B is any ball that contains the support of a. Therefore, $\cos(t\mathcal{D})a$ is in $L^1(M)$, and

$$
\int_M \cos(t\mathcal{D}) a \,\mathrm{d}\mu = \lim_{N\to\infty} \int_M \mathbf{1}_{B(c_B,N)}\, \cos(t\mathcal{D}) a \,\mathrm{d}\mu.
$$

Now, the last integral is the inner product $\big(\cos(t\mathcal{D})a, \mathbf{1}_{B(c_B,N)}\big)$ in $L^2(M)$, and is equal to $\big(a, \cos(t\mathcal{D})\mathbf{1}_{B(c_B,N)}\big)$, because $\cos(t\mathcal{D})$ is self adjoint. Observe that $\cos(t\mathcal{D})\mathbf{1}_{B(c_B,N)}$ is equal to $\cosh(\sqrt{b}t)$ on $B(c_B, N-t)$, because both functions are solutions of the wave equation $\partial_t^2 u + \mathcal{L}u = bu$ in $B(c_B, N) \times (0, \infty)$ and satisfy the same initial conditions $u(x, 0) = 1$, $\partial_t u(x, 0) = 0$ in $B(c_B, N)$. Hence, they coincide in $\{(x, t) : d(x, c_B) < N - t\}$, by standard energy estimates. If N is so big that $B(c_B, N - t)$ contains the support of a, then

$$
\big(a, \cos(t\mathcal{D})\mathbf{1}_{B(c_B,N)}\big) = \cosh(\sqrt{b}t) \int_M a \,\mathrm{d}\mu = 0,
$$

and (5.8) follows.

A straightforward consequence of (5.8) is that for any ν in $(-1/2,\infty)$ and for every H^1-atom a

$$\int_M \mathcal{J}_\nu(t\mathcal{D})a \,\mathrm{d}\mu = 0 \qquad \forall t \in \mathbb{R}^+. \tag{5.9}$$

Indeed,

$$\mathcal{J}_\nu(t\mathcal{D})a = \frac{\nu+2}{\sqrt{\pi}\,\Gamma(\nu+1/2)} \int_0^1 (1-s^2)^{\nu-1/2} \cos(st\mathcal{D})a \,\mathrm{d}s,$$

and the required conclusion follows from Fubini's Theorem. It is straightforward to check that similar considerations apply to the operator $\mathcal{D}_1$, so that for each ν in $[-1/2,\infty)$

$$\int_M \mathcal{J}_\nu(t\mathcal{D}_1)a \,\mathrm{d}\mu = 0 \qquad \forall t \in \mathbb{R}^+.$$

To prove (i) we just observe that

$$\begin{aligned}\int_M \mathcal{W}_\nu(\mathcal{D})a \,\mathrm{d}\mu &= \int_M \mathrm{d}\mu \int_{-\infty}^{\infty} w(t)\, \mathcal{J}_\nu(t\mathcal{D})a \,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \mathrm{d}t\, w(t) \int_M \mathcal{J}_\nu(t\mathcal{D})a \,\mathrm{d}\mu = 0,\end{aligned}$$

where the change of the order of integration is justified by Fubini's theorem.

Next we prove (ii). By (5.5), the function $S_0(\mathcal{D})a$ may be written as the sum of

$$(\widehat{\omega}_{\rho_0} * \widehat{K})(\mathcal{D})a \qquad \text{and} \qquad \sum_{j=1}^{k} c_{j,k} \int_{-\infty}^{\infty} K(t)\, \mathcal{O}^j \omega(\rho_0 t)\, \mathcal{O}^{k-j} \mathcal{J}_{k+1/2}(t\mathcal{D})a \,\mathrm{d}t,$$

where K is a compactly supported distribution on $\mathbb{R}$ such that $\widehat{K}$ is bounded and tK is in $L^\infty(\mathbb{R})$. It is a straightforward consequence of (i) that the integral of each summand of the sum above is equal to 0. Thus, to prove that the integral of $S_0(\mathcal{D})a$ is 0, it suffices to show that the integral of $(\widehat{\omega}_{\rho_0} * \widehat{K})(\mathcal{D})a$ makes sense and is equal to 0. Since $\widehat{K}$ is bounded, $\omega^\varepsilon \widehat{K}$ tends pointwise and boundedly to $\widehat{K}$ as ε tends to 0. Then $\widehat{\omega}_{\rho_0} * (\omega^\varepsilon \widehat{K})$ tends pointwise and boundedly to $\widehat{\omega}_{\rho_0} * \widehat{K}$ as ε tends to 0 by the Lebesgue dominated convergence theorem. Therefore the operator $\widehat{\omega}_{\rho_0} * (\omega^\varepsilon \widehat{K})(\mathcal{D})$ tends to the operator $\widehat{\omega}_{\rho_0} * \widehat{K}(\mathcal{D})$ in the strong operator topology of $L^2(M)$. Consequently $\widehat{\omega}_{\rho_0} * (\omega^\varepsilon \widehat{K})(\mathcal{D})a$ tends to $\widehat{\omega}_{\rho_0} * \widehat{K}(\mathcal{D})a$ in $L^2(M)$ as ε tends to 0.

Suppose that the support of a is contained in the ball B. Since the function $\omega^{\rho_0}\,(\widehat{\omega}_\varepsilon * K)$ is in $L^1(\mathbb{R})$,

$$\big[\widehat{\omega}_{\rho_0} * (\omega^\varepsilon \widehat{K})\big](\mathcal{D})a = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^{\rho_0}(t)\, (\widehat{\omega}_\varepsilon * K)(t)\, \cos(t\mathcal{D})a \,\mathrm{d}t.$$

Since the support of $\omega^{\rho_0}\,(\widehat{\omega}_\varepsilon * K)$ is contained in $[-1,1]$, all the functions $\big[\widehat{\omega}_{\rho_0} * (\omega^\varepsilon \widehat{K})\big](\mathcal{D})a$ are supported in the ball $B(c_B, r_B+1)$ by finite propagation speed, and

$$\int_M \big[\widehat{\omega}_{\rho_0} * (\omega^\varepsilon \widehat{K})\big](\mathcal{D})a \,\mathrm{d}\mu = 0$$

by (i). Thus, the function $\widehat{\omega}_{\rho_0} * \widehat{K}(\mathcal{D})a$ is also supported in $B(c_B, r_B+1)$. Hence $\widehat{\omega}_{\rho_0} * (\omega^\varepsilon \widehat{K})(\mathcal{D})a$ tends to $\widehat{\omega}_{\rho_0} * \widehat{K}(\mathcal{D})a$ in $L^1(M)$ as ε tends to 0, so that

$$\int_M (\widehat{\omega}_{\rho_0} * \widehat{K})(\mathcal{D})a \,\mathrm{d}\mu = \lim_{\varepsilon\to 0} \int_M \widehat{\omega}_{\rho_0} * (\omega^\varepsilon \widehat{K})(\mathcal{D})a \,\mathrm{d}\mu = 0,$$

as required to conclude the proof of (ii). $\square$

REMARK 5.6. Note that for every ν in $[-1/2, \infty)$ the function $\lambda \mapsto \mathcal{J}_\nu(t\lambda)$ is even and of entire of exponential type t, so that kernel $k_{\mathcal{J}_\nu(t\mathcal{D})}$ of the operator $\mathcal{J}_\nu(t\mathcal{D})$ is supported in the set $\{(x, y) \in M \times M : d(x, y) \le t\}$ by the finite propagation speed. A similar remark applies to the kernel of the operator $\mathcal{J}_\nu(t\mathcal{D}_1)$.

5.3. *Economical decomposition of atoms*

The following lemma produces an *economical* decomposition of atoms supported in "big" balls as finite linear combination of atoms supported in balls of radius at most 1, and is key to prove Theorem 3.4 below. The idea is "to transport charges along geodesics".

LEMMA 5.7. *There exists a constant C such that for every H^1-atom a supported in a ball B of radius $r_B > 1$*

$$\|a\|_{H^1} \le C\, r_B,$$

where $\|a\|_{H^1}$ is the atomic norm in $H^1(M)$ associated to the scale 1.

Proof. Denote by $\mathfrak{S}$ a 1/3-discretisation of M, i.e. a set of points in M that is maximal with respect to the property

$$\min\{d(z, w) : z, w \in \mathfrak{S}, z \ne w\} > 1/3, \quad \text{and} \quad d(\mathfrak{S}, x) \le 1/3 \qquad \forall x \in M.$$

The family $\{B(z, 1) : z \in \mathfrak{S}\}$ is a covering of M which is uniformly locally finite, by the uniform ball size and the locally doubling properties. By the same token, the set $B \cap \mathfrak{S}$ is finite and has at most N points $z_1, \ldots, z_N$, with $N \le C\,\mu(B)$, where C is a constant which does not depend on B. Denote by B_j the ball with centre z_j and radius 1, and by $\{\psi_j : j = 1, \ldots, N\}$ a partition of unity on B subordinated to the covering $\{B_j : j = 1, \ldots, N\}$.

Fix j in $\{1, \ldots, N\}$ and denote by $z_j^0, \ldots, z_j^{N_j}$ points on a minimizing geodesic joining z_j and c_B, with the property that $z_j^0 = z_j$, $z_j^{N_j} = c_B$, and $d(z_j^h, z_j^{h+1})$ is approximately equal to 1/3. Note that $N_j \le 4r_B$. Denote by B_j^h the ball $B(z_j^h, 1/12)$, for $j = 1, \ldots, N$ and $h = 0, \ldots, N_j$. Then the balls B_j^h are disjoint, $B_j^h \subset B(z_j^h, 1) \cap B(z_j^{h+1}, 1)$ and $B_j^{N_j} = B(c_B, 1/12)$.

Denote by ϕ_j^h a nonnegative function in $C_c^\infty(B_j^h)$ that has integral 1. By the uniform ball size property we may choose the functions ϕ_j^h so that there exists a constant A such that $\|\phi_j^h\|_2 \le A$ for all h and j.

Now, denote by a_j^0 the function $a\,\psi_j$. Clearly

$$a = \sum_{j=1}^N \psi_j\, a = \sum_{j=1}^N a_j^0.$$

Next, define

$$a_j^1 = a_j^0 - \phi_j^0 \int_M a_j^0 \,\mathrm{d}\mu \quad \text{and} \quad a_j^h = (\phi_j^{h-2} - \phi_j^{h-1}) \int_M a_j^0 \,\mathrm{d}\mu, \quad 2 \le h \le N_j + 1.$$

Then, for every h in $\{1,\dots,N_j\}$, the support of a_j^h is contained in $B(z_j^{h-1},1)$, the integral of a_j^h vanishes and

$$\begin{aligned}\|a_j^h\|_2 &\leq 2A\int_M |a_j^0|\,\mathrm{d}\mu \\ &\leq C\,\|a_j^0\|_2\,\mu(B_j)^{1/2} \\ &\leq C\,\|a_j^0\|_2\,\mu(B_j^h)^{-1/2}.\end{aligned}$$

In the last two inequalities we have used the fact that for each r in $\mathbb{R}^+$ the supremum of $\mu(B)$ over all balls B of radius r is finite by the uniform ball size property. Hence there exists a constant C, independent of j and h, such that

$$\|a_j^h\|_{H^1} \leq C\,\|a_j^0\|_2. \tag{5.10}$$

Moreover

$$a_j^0 = \sum_{h=1}^{N_j+1} a_j^h + \phi_j^{N_j}\int_M a_j^0\,\mathrm{d}\mu.$$

Thus

$$a = \sum_{j=1}^{N}\sum_{h=1}^{N_j+1} a_j^h,$$

because $\sum_j \int_M a_j^0\,\mathrm{d}\mu = \int_M a\,\mathrm{d}\mu = 0$ and all the functions $\phi_j^{N_j}$, $j=1,\dots,N_j$ coincide, for $B_j^{N_j} = B(c_B,1/12)$. Now we use (5.10) and the fact that $N_j \leq C\,r_B$, and conclude that

$$\begin{aligned}\|a\|_{H^1} &\leq C\sum_{j=1}^{N}\sum_{h=1}^{N_j+1}\|a_j^0\|_2 \\ &\leq C\,r_B\sum_{j=1}^{N}\|a_j^0\|_2.\end{aligned}$$

Then we use Schwarz's inequality and the fact that $N \leq C\,\mu(B)$, and obtain that

$$\begin{aligned}\|a\|_{H^1} &\leq C\,r_B\,N^{1/2}\Big(\sum_{j=1}^{N}\|a_j^0\|_2{}^2\Big)^{1/2} \\ &\leq C\,r_B\,\mu(B)^{1/2}\,\|a\|_2 \\ &\leq C\,r_B.\end{aligned}$$

The last inequality follows because a is a H^1-atom supported in the ball B.

This completes the proof of the lemma. □

5.4. *Proof of Theorem 3.4*

For the reader's convenience, we recall one of the properties of functions in $H^\infty(\mathbf{S}_W;J)$ (see Definition 3.3), which will be key in the proof of Theorem 3.4.

LEMMA 5.8 [**21**, Lemma 5.4]. *Suppose that J is an integer ≥ 2, and that W is in $\mathbb{R}^+$. Then there exists a positive constant C such that for every function f in $H^\infty\big(\mathbf{S}_W;J\big)$, and for every positive integer $h \leq J-2$*

$$|\mathcal{O}^h\widehat{f}(t)| \leq C\,\|f\|_{\mathbf{S}_W;J}\,|t|^{h-J}\,\mathrm{e}^{-W|t|} \qquad \forall t\in\mathbb{R}\setminus\{0\}.$$

We restate Theorem 3.4 for the reader's convenience.

THEOREM. **3.5** *Assume that α and β are as in (1.1), and δ as in (3.1). Denote by N the integer $[n/2+1]+1$. Suppose that J is an integer $> \max\big(N+2+\alpha/2-\delta, N+1/2\big)$. Then there exists a constant C such that*

$$|||m(\mathcal{D})|||_{H^1} \leq C\,\|m\|_{\mathbf{S}_\beta;J} \qquad \forall m \in H^\infty\big(\mathbf{S}_\beta;J\big).$$

Proof. For notational convenience, in this proof we shall write $\mathcal{J}$ instead of $\mathcal{J}_{N-1/2}$.

Step I: reduction of the problem. We claim that it suffices to prove that for each H^1-atom a the function $m(\mathcal{D})\,a$ may be written as the sum of atoms with supports contained in balls of $\mathcal{B}_1$, with ℓ^1 norm of the coefficients controlled by $C\,\|m\|_{\mathbf{S}_\beta;J}$.

Indeed, by arguing as in [**34**, Thm 4.1], we may then show that $m(\mathcal{D})$ extends to a bounded operator from $H^1(M)$ to $L^1(M)$, with norm dominated by $C\,\|m\|_{\mathbf{S}_\beta;J}$. Note that [**34**, Thm 4.1] is stated for spaces of homogeneous type. However, its proof extends to the present setting. Now, suppose that f is a function in $H^1(M)$ and that $f=\sum_j \lambda_j\, a_j$ is an atomic decomposition of f with $\|f\|_{H^1} \geq \sum_j |\lambda_j| - \varepsilon$. Then $m(\mathcal{D})f = \sum_j \lambda_j\, m(\mathcal{D})a_j$, where the series is convergent in $L^1(M)$, because $m(\mathcal{D})$ extends to a bounded operator from $H^1(M)$ to $L^1(M)$. But the partial sums of the series $\sum_j \lambda_j\, m(\mathcal{D})a_j$ is a Cauchy sequence in $H^1(M)$, hence the series is convergent in $H^1(M)$, and the sum must be the function $m(\mathcal{D})f$. Then

$$\begin{aligned}\|m(\mathcal{D})f\|_{H^1} &\leq \sum_j |\lambda_j|\,\|m(\mathcal{D})a_j\|_{H^1}\\ &\leq C\,\|m\|_{\mathbf{S}_\beta;J}\sum_j|\lambda_j|\\ &\leq C\,\|m\|_{\mathbf{S}_\beta;J}\,(\|f\|_{H^1}+\varepsilon),\end{aligned}$$

and the required conclusion follows by taking the infimum of both sides with respect to all admissible decompositions of f.

Step II: splitting of the operator. Let ω be the cut-off function defined in Section 3. Clearly $\widehat{\omega}*m$ and $m-\widehat{\omega}*m$ are bounded functions. Define the operators $\mathcal{S}$ and $\mathcal{T}$ spectrally by

$$\mathcal{S} = (\widehat{\omega}*m)(\mathcal{D}) \qquad \text{and} \qquad \mathcal{T} = (m-\widehat{\omega}*m)(\mathcal{D}).$$

Then $m(\mathcal{D}) = \mathcal{S}+\mathcal{T}$. We analyse the operators $\mathcal{S}$ and $\mathcal{T}$ in Step III and Step IV respectively.

Suppose that a is a H^1-atom supported in $B(p,R)$ for some p in M and $R\leq 1$.

Step III: analysis of $\mathcal{S}$. In the following, we shall need to estimate the $L^2(M)$ norm of the differential of the kernel of certain operators related to $\mathcal{S}$. To this end, and to be able to apply [**30**, Proposition 2.2 (iii)], we write the operator $\mathcal{S}$ as a function of the operator $\mathcal{D}_1$, rather than of $\mathcal{D}$. Recall that $\mathcal{D}_1 = \sqrt{\mathcal{D}^2+\kappa^2}$.

Since $\widehat{\omega}*m$ is an *even* entire function of exponential type 1, the function S, defined by

$$S(\zeta) = (\widehat{\omega}*m)\big(\sqrt{\zeta^2-\kappa^2}\big) \qquad \forall \zeta\in\mathbb{C},$$

is well defined, and is of exponential type 1. Hence its Fourier transform has support in $[-1,1]$. It is straightforward to check that

$$\mathcal{S} = S(\mathcal{D}_1),$$

and that

$$\|S\|_{\mathrm{Mih}(J)} \leq C\,\|\widehat{\omega}*m\|_{\mathrm{Mih}(J)},$$

where the constant C does not depend on m. By arguing much as in the proof of [**21**, Proposition 5.3], we may show that $\|\widehat{\omega} * m\|_{\mathrm{Mih}(J)} \leq C\,\|m\|_{\mathrm{Mih}(J)}$, where C is independent of m. Clearly

$$\|m\|_{\mathrm{Mih}(J)} \leq \|m\|_{\mathbf{S}_\beta;J} \qquad \forall m \in H^\infty(\mathbf{S}_\beta;J).$$

Hence there exists a constant C such that

$$\|S\|_{\mathrm{Mih}(J)} \leq C\,\|m\|_{\mathbf{S}_\beta;J} \qquad \forall m \in H^\infty(\mathbf{S}_\beta;J). \tag{5.11}$$

Define the functions S_i as in (5.5) and (5.6), but with $N-1$ in place of k and the Fourier transform of S in place of K. We further decompose $\mathcal{S}$ as $\sum_{i=0}^d S_i(\mathcal{D}_1)$, where d is as in (5.3). The function S_0 is bounded by Lemma 5.4 (iii), hence $S_0(\mathcal{D}_1)$ is bounded on $L^2(M)$ by the spectral theorem, and

$$|||S_0(\mathcal{D}_1)|||_2 \leq \|S_0\|_\infty \leq C\,\|S\|_{\mathrm{Mih}(2)} \leq C\,\|m\|_{\mathbf{S}_\beta;J}.$$

Observe that the support of the kernel of the operator $S_i(\mathcal{D}_1)$ is contained in $\{(x,y) : d(x,y) \leq 4^{i+1}R\}$ by the finite propagation speed. Thus the support of $S_i(\mathcal{D}_1)a$ is contained in the ball with centre p and radius $(4^{i+1}+1)R$, which henceforth we denote by B_i. In particular $S_0(\mathcal{D}_1)a$ is supported in $B_0 = B(p,5R)$, and

$$\|S_0(\mathcal{D}_1)a\|_2 \leq C\,|||S_0(\mathcal{D}_1)|||_2\,\|a\|_2 \leq C\,R^{-n/2}\,\|m\|_{\mathbf{S}_\beta;J}.$$

Furthermore, the integral of $S_0(\mathcal{D}_1)a$ vanishes by Proposition 5.5 (ii), so that $S_0(\mathcal{D}_1)\,a$ is a constant multiple of a H^1-atom.

Denote by $k_{S_i(\mathcal{D}_1)}$ the integral kernel of the operator $S_i(\mathcal{D}_1)$. Observe that

$$S_i(\mathcal{D}_1)\,a(x) = \int_{B(p,R)} a(y)\,\big[k_{S_i(\mathcal{D}_1)}(x,y) - k_{S_i(\mathcal{D}_1)}(x,p)\big]\,\mathrm{d}\mu(y).$$

By Minkowski's integral inequality and the fact that the support of $S_i(\mathcal{D}_1)\,a$ is contained in B_i, we have that

$$\begin{aligned}\|S_i(\mathcal{D}_1)\,a\|_2 &= \|S_i(\mathcal{D}_1)\,a\|_{L^2(B_i)} \\ &\leq \int_{B(p,R)} |a(y)|\,I_i(y)\,\mathrm{d}\mu(y),\end{aligned}$$

where

$$I_i(y) = \|k_{S_i(\mathcal{D}_1)}(\cdot,y) - k_{S_i(\mathcal{D}_1)}(\cdot,p)\|_{L^2(B_i)} \qquad \forall y \in B(p,R).$$

To estimate $I_i(y)$, we observe that

$$I_i(y) \leq d(y,p)\,\sup_{z\in M}\,\big\|\mathrm{d}_2 k_{S_i(\mathcal{D}_1)}(\cdot,z)\big\|_{L^2(B_i)}$$

and, by Lemma 5.4 (ii) (with $k = N-1$),

$$\mathrm{d}_2 k_{S_i(\mathcal{D}_1)}(\cdot,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi^{\rho_i}(t)\,P_N(\mathcal{O})\widehat{S}(t)\;\mathrm{d}_2 k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,z)\,\mathrm{d}t.$$

Recall that ϕ^{ρ_i} is supported in $E_i = \{t \in \mathbb{R} : 4^{i-1}R \leq |t| \leq 4^{i+1}R\}$, that the support of $\widehat{S}$ is contained in $[-1,1]$ and that $d(p,y) < R$. Then, by [**30**, Proposition 2.2 (ii)] (with $\mathcal{J}$ in place of F), there exists a constant C, independent of i and R, such that

$$\begin{aligned}I_i(y) &\leq C\,d(y,p)\int_{-\infty}^{\infty} \phi^{\rho_i}(t)\,|P_N\mathcal{O})\widehat{S}(t)|\;\sup_{z\in M}\,\big\|\mathrm{d}_2 k_{\mathcal{J}(t\mathcal{D}_1)}(\cdot,z)\big\|_{L^2(B_i)}\;\mathrm{d}t \\ &\leq C\,\|tP_N(\mathcal{O})\widehat{S}\|_\infty\,R\int_{E_i} |t|^{-n/2-2}\,\mathrm{d}t \\ &\leq C\,\|m\|_{\mathbf{S}_\beta;J}\,R\,(4^iR)^{-n/2-1}\,.\end{aligned}$$

Thus,

$$\begin{aligned}\|S_i(\mathcal{D}_1)\, a\|_2 &\leq C\ \|m\|_{\mathbf{S}_\beta;J}\, 4^{-i}\, (4^i R)^{-n/2}\, \|a\|_1 \\ &\leq C\ \|m\|_{\mathbf{S}_\beta;J}\, 4^{-i}\, \mu(B_i)^{-1/2}.\end{aligned}$$

Furthermore the integral of $S_i(\mathcal{D}_1)\, a$ vanishes by Proposition 5.5 (i), so that the function $4^i\, S_i(\mathcal{D}_1)\, a$ is a constant multiple of a H^1-atom. Thus

$$\begin{aligned}\|\mathcal{S}\, a\|_{H^1} &\leq C\ \|m\|_{\mathbf{S}_\beta;J} \sum_{i=0}^{\infty} 4^{-i} \\ &\leq C\, \|m\|_{\mathbf{S}_\beta;J}.\end{aligned}$$

Step IV: analysis of $\mathcal{T}$. For each j in $\{1, 2, 3, \ldots\}$, define ω_j by the formula

$$\omega_j(t) = \omega(t-j) + \omega(t+j) \qquad \forall t \in \mathbb{R}. \tag{5.12}$$

Observe that $\sum_{j=1}^{\infty} \omega_j = 1 - \omega$ and that the support of ω_j is contained in the set of all t in $\mathbb{R}$ such that $j - 3/4 \leq |t| \leq j + 3/4$.

Since m is in $H^\infty\big(\mathbf{S}_\beta; J\big)$ and $J \geq N+2$, the function $\widehat{m}$ and its derivatives up to the order N are rapidly decreasing at infinity by Lemma 5.8, so that $\mathcal{O}^\ell(\omega_j\, \widehat{m})$ is in $L^1(\mathbb{R}) \cap C_0(\mathbb{R}^+)$ for all ℓ in $\{0, \ldots, N\}$, and so does $P_N(\mathcal{O})(\omega_j\, \widehat{m})$. In the rest of this proof, we write $\Omega_{j,N}$ instead of $P_N(\mathcal{O})(\omega_j\, \widehat{m})$. Observe that the support of $\Omega_{j,N}$ is contained in $\{t \in \mathbb{R} : j - 3/4 \leq |t| \leq j + 3/4\}$.

Define the function $T_j : \mathbb{R} \to \mathbb{C}$ by

$$T_j(\lambda) = \int_{-\infty}^{\infty} \Omega_{j,N}(t)\, \mathcal{J}(t\lambda)\, \mathrm{d}t \qquad \forall \lambda \in \mathbb{R}. \tag{5.13}$$

We may use the observation that $(m - \widehat{\omega} * m)\widehat{\ } = \sum_{j=1}^{\infty} \omega_j\, \widehat{m}$ and formula (5.2), and write

$$\begin{aligned}(m - \widehat{\omega} * m)(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \big(1 - \omega(t)\big)\, \widehat{m}(t)\, \cos(t\lambda)\, \mathrm{d}t \\ &= \sum_{j=1}^{\infty} T_j(\lambda).\end{aligned}$$

Then, by the spectral theorem,

$$\mathcal{T} a = \sum_{j=1}^{\infty} T_j(\mathcal{D}) a.$$

By the asymptotics of $J_{N-1/2}$ [**28**, formula (5.11.6), p. 122]

$$\sup_{s>0} |(1+s)^N\, \mathcal{J}(s)| < \infty.$$

Since $N - 1/2 > (n+1)/2$, we may apply [**30**, Proposition 2.2 (i)] and conclude that

$$\begin{aligned}\|\mathcal{J}(t\mathcal{D})a\|_2 &\leq \|a\|_1\, \big|\!\big|\!\big|\mathcal{J}(t\mathcal{D})\big|\!\big|\!\big|_{1;2} \\ &\leq \sup_{y \in M}\, \big\|k_{\mathcal{J}(t\mathcal{D})}(\cdot, y)\big\|_2 \\ &\leq C\, |t|^{-n/2}\, \big(1 + |t|\big)^{n/2 - \delta} \qquad \forall t \in \mathbb{R} \setminus \{0\}.\end{aligned}$$

Then $\mathcal{J}(t\mathcal{D})a$ is supported in $B(p,t+R)$, and has integral 0 by Proposition 5.5 (i). Observe that

$$\begin{aligned}\|T_j(\mathcal{D})a\|_2 &\leq C\int_{-\infty}^{\infty}|\Omega_{j,N}(t)|\,\|\mathcal{J}(t\mathcal{D})a\|_2\,\mathrm{d}t\\ &\leq C\int_{j-3/4}^{j+3/4}|\Omega_{j,N}(t)|\,|t|^{-n/2}\big(1+|t|\big)^{n/2-\delta}\,\mathrm{d}t\\ &\leq C\,\|m\|_{\mathbf{S}_\beta;J}\;j^{N-J-\delta}\,\mathrm{e}^{-\beta j}\qquad\forall j\in\{1,2,\ldots\}.\end{aligned}\tag{5.14}$$

In the last inequality we have used Lemma 5.8 and [**30**, Proposition 2.2 (i)]. Note that $j^{\delta+J-N-\alpha/2}\,T_j(\mathcal{D})a$ is a constant multiple of a H^1-atom. Indeed, $T_j(\mathcal{D})a$ is a function in $L^2(M)$ with support contained in $B\big(p,j+1\big)$, and has integral 0 by Proposition 5.5 (i). Moreover

$$\begin{aligned}\|j^{\delta+J-N-\alpha/2}\,T_j(\mathcal{D})a\|_2 &\leq C\,\|m\|_{\mathbf{S}_\beta;J}\;j^{-\alpha/2}\,\mathrm{e}^{-\beta j}\\ &\leq C\,\|m\|_{\mathbf{S}_\beta;J}\;\mu\big(B(p,j+1)\big)^{-1/2}\qquad\forall j\in\{1,2,\ldots\}.\end{aligned}$$

Hence we may write

$$\mathcal{T}a=\sum_{j=1}^{\infty}\lambda_j\,a_j',$$

where a_j' is a H^1-atom supported in $B\big(p,j+1\big)$, and

$$\lambda_j=C\,\|m\|_{\mathbf{S}_\beta;J}\,j^{N+\alpha/2-J-\delta}.$$

By Lemma 5.7 we have $\|a_j'\|_{H^1}\leq C\,j$, so that

$$\begin{aligned}\|\mathcal{T}a\|_{H^1} &\leq \sum_{j=1}^{\infty}|\lambda_j|\,\|a_j'\|_{H^1}\\ &\leq C\,\|m\|_{\mathbf{S}_\beta;J}\;\sum_{j=1}^{\infty}j^{1+N+\alpha/2-J-\delta},\end{aligned}$$

which is finite (and independent of a) because $J>2+N+\alpha/2-\delta$.

Step V: conclusion. By Step III and Step IV there exists a constant C such that for every H^1-atom a with support contained in a ball of radius at most 1

$$\|\mathcal{S}a\|_{H^1}+\|\mathcal{T}a\|_{H^1}\leq C\,\|m\|_{\mathbf{S}_\beta;J}.$$

Then Step II implies that

$$\|m(\mathcal{D})a\|_{H^1}\leq C\,\|m\|_{\mathbf{S}_\beta;J}.$$

The required conclusion follows from Step I. □

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