PALEY–WIENER THEOREMS FOR THE U(n)–SPHERICAL TRANSFORM ON THE HEISENBERG GROUP

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ABSTRACT. We prove several Paley–Wiener-type theorems related to the spherical transform on the Gelfand pair $\big(H_n \rtimes \mathrm{U}(n), \mathrm{U}(n)\big)$, where H_n is the $2n+1$-dimensional Heisenberg group.

Adopting the standard realization of the Gelfand spectrum as the Heisenberg fan in $\mathbb{R}^2$, we prove that spherical transforms of $\mathrm{U}(n)$–invariant functions and distributions with compact support in H_n admit a unique entire extension to $\mathbb{C}^2$, and we find real-variable characterizations of such transforms. Next, we characterize the inverse spherical transforms of compactly supported functions and distributions on the fan, giving analogous characterizations. This requires a preliminary analysis of spherical transforms of $\mathrm{U}(n)$-invariant tempered distributions, which are identified as distributions on $\mathbb{R}^2$ supported on the fan which are synthetizable, i.e., vanishing on functions which are zero on the fan.

1. INTRODUCTION

The spherical transform for the Gelfand pair $\big(H_n \rtimes \mathrm{U}(n), \mathrm{U}(n)\big)$ maps $\mathrm{U}(n)$–invariant functions, i.e. radial functions, on the Heisenberg group H_n to functions on the Heisenberg fan Σ, which is naturally realized as a closed subset of $\mathbb{R}^2$. In [4, 5] we have studied the image of the space $\mathcal{S}_{\mathrm{rad}}(H_n)$ of radial Schwartz functions, showing that it consists of the restrictions to Σ of Schwartz functions on $\mathbb{R}^2$.

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In this paper we first use this result to extend the notion of spherical transform to tempered radial distributions, identifying such transforms as "synthetizable" distributions on $\mathbb{R}^2$ supported on Σ, i.e., vanishing on functions which are zero on the fan. Then we prove Paley–Wiener type theorems for the spherical transform $\mathcal{G}$ and its inverse.

The natural starting point is the extension of the domain of spherical transforms from the set of bounded spherical functions (the Gelfand spectrum) to the set of all spherical functions. Spherical functions are parametrized by the pairs $(\xi, \lambda) \in \mathbb{C}^2$ of their eigenvalues with respect to the two fundamental differential operators, L (the sublaplacian) and $i^{-1}T$ (the central derivative). In fact, for each (ξ, λ) there is a corresponding spherical function $\Phi_{\xi,\lambda}$, which depends on the eigenvalues in a holomorphic way. This allows to extend the spherical transform of functions and distributions with compact support to an entire function on $\mathbb{C}^2$.

Symmetrically, each spherical function $\Phi_{\xi,\lambda}$ extends to an entire function on the complexification of H_n, and the inversion formula shows that if the spherical transform of a function has compact support in the Gelfand spectrum, the function itself extends to an entire function.

For a radial distribution Λ with compact support on H_n, we give a "real-variable" characterization of the (entire) functions $\mathcal{G}\Lambda$ obtained in this way, which is based on estimates on Σ involving the operators $M_\pm$ of Benson and Ratcliff [8]. Such a real-variable version of the Paley–Wiener theorem is in the spirit of the works of Bang [7] and Tuan [25], later expanded and refined by Andersen and deJeu [1], for the Fourier transform in $\mathbb{R}^n$. We do not exclude that a "complex variable" characterization is possible, but we could not see any natural one.

Similar characterizations are obtained for the spherical transforms of radial L^2 functions and radial Schwartz functions. We take this opportunity for remarking that the entire extension of the transform of a function in $\mathcal{D}_{\mathrm{rad}}(H_n)$ needs not be Schwartz on $\mathbb{R}^2$. This shows that the Schwartz extension to $\mathbb{R}^2$ according to [5] can be different from the entire extension discussed here.

In the second part of the paper, we reverse the rôle of the two sides of the spherical transform and discuss the properties of the inverse spherical transform of functions and distributions with compact support in Σ. The results can be interpreted as concerning spectral Paley–Wiener theorems related to the joint spectral measure of a pair of differential operators, namely the sublaplacian and the central derivative.

For a synthetizable distribution U on $\mathbb{R}^2$ with compact support contained in Σ, we prove that its inverse spherical transform $\mathcal{G}^{-1}U$ is a function on the Heisenberg group $H_n \simeq \mathbb{R}^{2n+1}$ which admits an entire extension F to $\mathbb{C}^{2n+1}$. This part of the paper presents some closeness to the results of Fuhr [16], where a Paley–Wiener space is defined on H_1 as the image in L^2 of the spectral projection of the sublaplacian associated to the specific interval $[0, 2\pi]$. The author proves that, in contrast with the Euclidean case, the Paley–Wiener space cannot be characterized in terms of growth conditions. However we show that, for radial functions and synthetizable distributions supported on a compact subset of the spectrum, a "real-variable" characterization of their inverse spherical transforms is possible, of the same nature of the one given for the direct transform.

There is a wide literature on Paley–Wiener theorems on the Heisenberg group. The earliest result is due to Ando [2], followed by Thangavelu [21, 22, 23], Arnal and Ludwig [3], Narayanan and Thangavelu [19]. Results are mostly related to the group (operator-valued) Fourier transform and its inverse, but there are also "spectral" Paley–Wiener theorems, as in the already mentioned paper [16], where the condition of compact support on the transform of a given function is replaced by the condition that the function itself belongs to the image of the spectral measure of a compact set in $\mathbb{R}^+$ associated to the sublaplacian (see also Strichartz [20], Bray [10], and Dann and Ólafsson [12] in other contexts).

Our paper is organized as follows. In Section 2 we introduce the basic notation, in Section 3 we treat spherical functions noting that they can be extended to holomorphic functions in each variable and providing some easy estimates. Section 4 and Section 5 deal with the spherical transform of radial functions and radial tempered distributions, respectively. In Section 6 we prove some properties of the operators $M_\pm$ first introduced in [8]. These are exploited in Sections 7 and 8 to obtain real Paley–Wiener Theorems for the spherical transform and its inverse, respectively.

2. Notation

We denote by H_n the Heisenberg group, i.e., the real manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(z,t)(w,u) = \big(z+w, t+u+\tfrac{1}{2}\operatorname{Im}\langle w|z\rangle\big) \qquad \forall z,w\in\mathbb{C}^n, \quad t,u\in\mathbb{R},$$

where $\langle\cdot|\cdot\rangle$ denotes the Hermitian innner product in $\mathbb{C}^n$.

It is easy to check that the Lebesgue measure $dm = dz\,dt$ is a Haar measure on H_n.

We denote by T, Z_j and $\bar{Z}_j$, where $j=1,\dots,n$, the left-invariant vector fields

$$Z_j = \partial_{z_j} - \tfrac{i}{4}\,\bar{z}_j\,\partial_t \qquad \bar{Z}_j = \partial_{\bar{z}_j} + \tfrac{i}{4}\,z_j\,\partial_t, \qquad T = \partial_t\ .$$

The only nontrivial brackets are $T = -2i\,[Z_j, \bar{Z}_j]$.

The operators Z_j and $\bar{Z}_j$ are homogeneous of degree 1 while T is homogeneous of degree 2 with respect to the anisotropic dilations $r\cdot(z,t) = (rz, r^2t)$, where $r>0$ and $(z,t)\in H_n$. Let $I = (i_1,\dots,i_{2n+1})$ be in $\mathbb{N}^{2n+1}$; we denote by D^I a differential operator of homogeneous degree $\deg I = i_1 + \cdots + i_{2n} + 2i_{2n+1}$ of the form

$$D^I = Z_1^{i_1} \bar{Z}_1^{i_2} \cdots Z_n^{i_{2n-1}} \bar{Z}_n^{i_{2n}} T^{i_{2n+1}}. \tag{2.1}$$

The monomials D^I with $\deg I = j$ form a basis of the space of all left-invariant differential operators on H_n which are homogeneous of degree j.

We write $\mathcal{S}(H_n)$ for the Schwartz space of functions on H_n, i.e., the space of infinitely differentiable functions f on H_n such that all partial derivatives $D^I f$ of f are rapidly decreasing. The Schwartz space is equipped with the following family of norms, parametrized by a nonnegative integer p:

$$\|f\|_{(p)} = \sup_{(z,t)\in H_n} \left\{ (1 + |(z,t)|)^p \left| D^I f(z,t) \right| \,:\, \deg I \le p \right\} ,$$

where

$$|(z,t)| = \left(\frac{|z|^4}{16} + t^2 \right)^{1/4} .$$

We also define $\mathcal{A}$ as

$$\mathcal{A}(z,t) = \frac{|z|^2}{4} + it \qquad \forall (z,t) \in H_n,$$

so that $|A(z,t)| = |(z,t)|^2$.

3. Spherical functions

The unitary group $\mathrm{U}(n)$ acts on H_n via

$$k \cdot (z,t) = (kz, t) \qquad \forall (z,t) \in H_n, \quad k \in \mathrm{U}(n).$$

This action induces an action on functions f on H_n by the formula

$$k \cdot f(z,t) = f(k^{-1}z, t) \qquad \forall k \in \mathrm{U}(n), \quad (z,t) \in H_n.$$

We note that a function f on H_n is $\mathrm{U}(n)$–invariant if and only if it depends only on $|z|$ and t, therefore we shall call it radial. We denote by $\mathcal{S}_{\mathrm{rad}}(H_n)$ the space of radial Schwartz functions.

Denote by G the group given by the semidirect product $H_n \rtimes \mathrm{U}(n)$. We may identify the space of smooth bi-$\mathrm{U}(n)$–invariant functions $\mathcal{D}(G//\mathrm{U}(n))$ with the algebra $\mathcal{D}_{\mathrm{rad}}(H_n)$ of smooth radial functions on H_n with compact support. It is known [17, 11] that $(G, \mathrm{U}(n))$ is a Gelfand pair, i.e., $\mathcal{D}_{\mathrm{rad}}(H_n)$ is a commutative algebra. We may also identify the commutative algebra $\mathbb{D}(G/\mathrm{U}(n))$ of G–invariant differential operators on $G/\mathrm{U}(n)$ with the algebra $\mathbb{D}_{\mathrm{rad}}$ of all left-invariant and $\mathrm{U}(n)$–invariant differential operators on H_n, which has two essentially self-adjoint generators, namely $i^{-1}T$ and the sublaplacian

$$L = -2\sum_{j=1}^{n}\big(Z_j\bar{Z}_j + \bar{Z}_j Z_j\big).$$

The spherical functions are characterized as the joint eigenfunctions of all G–invariant differential operators on $G/\mathrm{U}(n)$, i.e., as the radial eigenfunctions of $i^{-1}T$ and L, normalized to take value 1 at identity. Spherical functions are analytic and are uniquely determined by the pair $(\xi, \lambda) \in \mathbb{C}^2$ of their eigenvalues relative to L and $i^{-1}T$ respectively.

The next subsection shows that the spherical function $\Phi_{\xi,\lambda}$ exists for every pair (ξ, λ) of eigenvalues and that depends holomorphically on variables and parameters.

3.1. **Holomorphy of spherical functions.** We initially consider real eigenvalues ξ and λ and look for a radial solution of the system

$$\begin{cases} Lu = \xi\, u \\ Tu = i\lambda\, u \\ u(0,0) = 1\ . \end{cases} \tag{3.1}$$

For $\lambda \neq 0$ one writes the solution in the form

$$u(z,t) = e^{i\lambda t}\, e^{-\lambda |z|^2/4}\, v(\lambda |z|^2/2),$$

obtaining that v satisfies the confluent hypergeometric differential equation

$$s\, v''(s) + (c-s)v'(s) - a\, v(s) = 0 \tag{3.2}$$

with parameters $a = \frac{n}{2} - \frac{\xi}{2\lambda}$, $c = n$. The normalized solution of (3.2) is the confluent hypergeometric function

$${}_1F_1(a,c;s) = 1 + \frac{a}{c}\, s + \frac{a(a+1)}{c(c+1)}\, \frac{s^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\, \frac{s^k}{k!}\ ,$$

where $(a)_0 = 1$, $(a)_k = \Gamma(a+k)/\Gamma(a)$, so that for real $\lambda \neq 0$

$$u(z,t) = e^{i\lambda t}\, e^{-\lambda |z|^2/4}\, {}_1F_1(\tfrac{n}{2} - \tfrac{\xi}{2\lambda}, n; \lambda |z|^2/2)\ .$$

When $\lambda = 0$ and ξ is real, a similar procedure shows that

$$u(z,t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n)_k}\, \frac{(\xi |z|^2/4)^k}{k!} = \mathcal{J}_{n-1}(\xi |z|^2/4) \qquad \forall (z,t) \in H_n,$$

where

$$\mathcal{J}_\beta(s) = \sum_{k=0}^{+\infty} \frac{(-s)^k}{k!\,(\beta+1)_k} \qquad \forall s \in \mathbb{C}.$$

Note that $J_\beta(u) = \dfrac{(u/2)^\beta}{\beta!}\mathcal{J}_\beta\left(u^2/4\right)$ is the Bessel function of the first kind of order β.

Therefore for every pair of real numbers ξ and λ we have the spherical function

$$\Phi_{\xi,\lambda}(z,t) = \begin{cases} e^{i\lambda t}\, e^{-\lambda |z|^2/4}\, {}_1F_1\left(\frac{n}{2} - \frac{\xi}{2\lambda}, n; \frac{\lambda |z|^2}{2}\right) & \lambda \neq 0 \\ \mathcal{J}_{n-1}(\xi |z|^2/4) & \lambda = 0 \end{cases} \qquad \forall (z,t) \in H_n.$$

We now verify that $\lambda \longmapsto \Phi_{\xi,\lambda}(z,t)$ is regular in $\lambda = 0$. Indeed, something more holds.

Lemma 3.1. *The function* $(x, y, t, \xi, \lambda) \longmapsto \Phi_{\xi,\lambda}(x+iy, t)$ *extends to a holomorphic function on* $\mathbb{C}^{2n+3}$.

Proof. Note that when $\lambda \neq 0$, $z = x + iy$,

$$
\begin{aligned}
{}_1F_1\left(\frac{n\lambda - \xi}{2\lambda}, n; \frac{\lambda |z|^2}{2}\right) &= \sum_{k=0}^{\infty} \frac{\left(\frac{n\lambda-\xi}{2\lambda}\right)_k}{(n)_k\, k!} \left(\frac{\lambda|z|^2}{2}\right)^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(x^2+y^2)^k}{(n)_k\, k!\, 4^k} \prod_{d=0}^{k-1} \left(\lambda(2d+n) - \xi\right),
\end{aligned}
$$

so that for all (ξ, λ, x, y, t) the function

$$
\Phi_{\xi,\lambda}(x+iy, t) = e^{i\lambda t}\, e^{-\lambda(x^2+y^2)/4} \left(1 + \sum_{k=1}^{\infty} \frac{(x^2+y^2)^k}{(n)_k\, k!\, 4^k} \prod_{d=0}^{k-1} \left(\lambda(2d+n) - \xi\right)\right) \tag{3.3}
$$

is a series of entire functions converging uniformly on compact sets. □

By analytic continuation, $\Phi_{\xi,\lambda}$ is the spherical function for every $(\xi, \lambda) \in \mathbb{C}^2$.

3.2. **Bounded spherical functions.** The Gelfand spectrum of the Banach algebra $L^1_{\mathrm{rad}}(H_n)$ of radial integrable functions is given by the set of normalized bounded spherical functions, equipped with the compact open topology. We recall [18] that $\Phi_{\xi,\lambda}$ is bounded on H_n if and only if (ξ, λ) belongs to the so called Heisenberg fan given by

$$
\Sigma = \Sigma^* \cup \left\{(\xi, 0) \in \mathbb{R}^2 \ : \ \xi \geq 0\right\},
$$

where

$$
\Sigma^* = \left\{(\xi, \lambda) \in \mathbb{R}^2 : \lambda \neq 0\,,\, \xi = |\lambda|(2j+n)\,,\ j \in \mathbb{N}\right\} .
$$

It is known that Σ is homeomorphic to the Gelfand spectrum [14, 9].

When (ξ, λ) is in Σ^*, the spherical function $\Phi_{\xi,\lambda}$ can be written in terms of Laguerre polynomials, which is the form that we usually find in literature. Indeed, the relation (see

[13, p. 253, formula (7)]

$$ {}_1F_1(a,n;x) = e^x\, {}_1F_1(n-a,n;-x), \tag{3.4} $$

implies that

$$ \Phi_{\xi,\lambda}(z,0) = \Phi_{\xi,|\lambda|}(z,0) = e^{-|\lambda||z|^2/4}\, {}_1F_1\left(\frac{n}{2} - \frac{\xi}{2|\lambda|}, n; \frac{|\lambda||z|^2}{2}\right) $$

and when $\xi = |\lambda|(2j+n)$ the hypergeometric function in the previous formula coincides with the normalized j^{th} Laguerre polynomial of order $n-1$, i.e.,

$$ {}_1F_1\left(-j, n; \frac{|\lambda||z|^2}{2}\right) = \frac{1}{\binom{j+n-1}{j}} \sum_{k=0}^{j} \binom{j+\beta}{j-k} \frac{(-\frac{|\lambda||z|^2}{2})^k}{k!}. $$

3.3. **Estimates of derivatives of spherical functions.** In this subsection we exploit the fact that the spherical functions $\{\Phi_{\xi,\lambda}\}_{(\xi,\lambda)\in\Sigma}$ are averages of coefficients of irreducible unitary representations of H_n to give some estimates that we shall need in the sequel. Referring to the Bargmann-Fock model of the irreducible representations π^λ of H_n associated to the character $e^{i\lambda t}$ on the center, we represent the operators $\pi^\lambda(z,t)$ as matrices $\big(\pi^\lambda_{\mathbf{j},\mathbf{k}}(z,t)\big)_{\mathbf{j},\mathbf{k}\in\mathbb{N}^n}$ in the basis of normalized monomials (for more details see the monographs [15] or [24]). Then the bounded spherical functions can be written as averages of diagonal entries of this matrix according to the rule

$$ \Phi_{\xi,\lambda} = \frac{1}{\binom{j+n-1}{j}} \sum_{j=|\mathbf{j}|} \pi^\lambda_{\mathbf{j},\mathbf{j}} \qquad \text{where } \xi = |\lambda|(2j+n). \tag{3.5} $$

Lemma 3.2. *Let D^I be a differential operator of homogeneous degree $\deg I = \alpha$ as in (2.1). Then*

$$ |D^I\Phi_{\xi,\lambda}(z,t)| \le C_\alpha\,(1+\xi)^{\alpha/2} \qquad \forall(\xi,\lambda)\in\Sigma,\ (z,t)\in H_n. $$

Proof. Let $\lambda \neq 0$. Here and afterwards, if any component of the multiindeces $\mathbf{j}$ or $\mathbf{k}$ is negative, then $\pi^\lambda_{\mathbf{j},\mathbf{k}} = 0$. Since the representations are unitary, $|\pi^\lambda_{\mathbf{j},\mathbf{k}}| \leq 1$ and it is easy to check that

$$Z_i \pi^\lambda_{\mathbf{j},\mathbf{k}} = \begin{cases} -\sqrt{\frac{k_i\lambda}{2}}\,\pi^\lambda_{\mathbf{j},\mathbf{k}-\mathbf{e}_i} & \lambda > 0 \\ \sqrt{\frac{(k_i+1)|\lambda|}{2}}\,\pi^\lambda_{\mathbf{j},\mathbf{k}+\mathbf{e}_i} & \lambda < 0 \end{cases} \qquad \bar{Z}_i \pi^\lambda_{\mathbf{j},\mathbf{k}} = \begin{cases} \sqrt{\frac{(k_i+1)\lambda}{2}}\,\pi^\lambda_{\mathbf{j},\mathbf{k}+\mathbf{e}_i} & \lambda > 0 \\ -\sqrt{\frac{k_i|\lambda|}{2}}\,\pi^\lambda_{\mathbf{j},\mathbf{k}-\mathbf{e}_i} & \lambda < 0 \end{cases}$$

and $T\pi^\lambda_{\mathbf{j},\mathbf{j}} = i\,\lambda\,\pi^\lambda_{\mathbf{j},\mathbf{j}}$. Here $\mathbf{e}_i$ is the multiindex with just the i^{th} component equal to 1.

Suppose that (ξ, λ) is in Σ^*, with $\xi = |\lambda|(2|\mathbf{j}| + n)$. Therefore if $D^I = Z^{\mathbf{k}}\bar{Z}^{\mathbf{h}}T^s$ with $\deg I = \alpha = |\mathbf{k}| + |\mathbf{h}| + 2s$ we have

$$\begin{aligned} |D^I \pi^\lambda_{\mathbf{j},\mathbf{j}}| &= |\lambda|^s\,|Z^{\mathbf{k}}\bar{Z}^{\mathbf{h}}\pi^\lambda_{\mathbf{j},\mathbf{j}}| \\ &= |\lambda|^s \begin{cases} \sqrt{\prod_{i=1}^n \prod_{\ell=1}^{h_i} \frac{\lambda}{2}(j_i+\ell)}\,\sqrt{\prod_{i=1}^n \prod_{\ell=1}^{k_i} \frac{\lambda}{2}(j_i+h_i+1-\ell)}\,|\pi^\lambda_{\mathbf{j},\mathbf{j}+\mathbf{h}-\mathbf{k}}| & \lambda > 0 \\ \sqrt{\prod_{i=1}^n \prod_{\ell=1}^{h_i} \frac{|\lambda|}{2}(j_i+1-\ell)}\,\sqrt{\prod_{i=1}^n \prod_{\ell=1}^{k_i} \frac{|\lambda|}{2}(j_i-h_i+\ell)}\,|\pi^\lambda_{\mathbf{j},\mathbf{j}-\mathbf{h}+\mathbf{k}}| & \lambda < 0 \end{cases} \\ &\leq C\,|\lambda|^s \sqrt{|\lambda|^{|\mathbf{h}|+|\mathbf{k}|}(|\mathbf{j}|+|\mathbf{h}|)^{|\mathbf{h}|}\;\;(|\mathbf{j}|+|\mathbf{h}|+|\mathbf{k}|)^{|\mathbf{k}|}} \\ &\leq C_\alpha\,(1+\xi)^{\alpha/2}. \end{aligned}$$

Here we have used the fact that in Σ^* we have $\xi = |\lambda|(2|\mathbf{j}| + m) \geq |\lambda|$. By (3.5) the thesis follows on Σ^*, and by continuity the same estimates hold on Σ, thus proving the lemma. □

4. Spherical transform

As usual, we denote by $\langle \cdot, \cdot \rangle_{H_n}$ the dual pairing on H_n and we shall also write

$$\langle f, g \rangle_{H_n} = \int_{H_n} f(z,t)\,g(z,t)\,dz\,dt, \qquad \forall f, g \in \mathcal{S}(H_n).$$

Given a measurable function f on H_n we denote by $\check{f}$ the function defined by $\check{f}(x) = f(x^{-1})$ for every x in H_n.

4.1. **Definitions and main facts.** Let f be in $L^1_{\mathrm{rad}}(H_n)$. We define its spherical transform $\mathcal{G}f$ by

$$\mathcal{G}f(\xi,\lambda) = \int_{H_n} f(x)\,\Phi_{\xi,\lambda}(x^{-1})\,dx = \langle f, \check{\Phi}_{\xi,\lambda}\rangle_{H_n} \qquad \forall(\xi,\lambda)\in\Sigma.$$

Then $\mathcal{G}f$ is a continuous function on Σ.

The inversion formula for a function f in $\mathcal{S}_{\mathrm{rad}}(H_n)$ is

$$f(x) = \int_{\Sigma} \mathcal{G}f(\xi,\lambda)\,\Phi_{\xi,\lambda}(x)\,d\mu(\xi,\lambda) \qquad \forall x\in H_n,$$

where μ is the Plancherel measure defined by

$$\int_{\Sigma}\psi\,d\mu = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}}\sum_{j=0}^{\infty}\binom{j+n-1}{j}\psi(|\lambda|(2j+n),\lambda)\,|\lambda|^n\,d\lambda \qquad \forall\psi\in C_c(\Sigma).$$

It is easy to check that the function $(\xi,\lambda)\mapsto(1+\xi)^{-(n+2)}$ is in $L^1(\Sigma)$, so

$$\|\psi\|_{L^1(\Sigma)} \le C\,\|(1+|\xi|)^{n+2}\,\psi\|_{L^\infty(\mathbb{R}^2)} \qquad \forall\psi\in\mathcal{S}(\mathbb{R}^2). \tag{4.1}$$

As in [4], we denote by $\mathcal{S}(\Sigma)$ the space of restrictions to Σ of Schwartz functions on $\mathbb{R}^2$, endowed with the quotient topology $\mathcal{S}(\mathbb{R}^2)/\{\phi : \phi|_\Sigma = 0\}$. For radial Schwartz functions on H_n, we have proved the following result.

Theorem 4.1. [4, Corollary 1.2] *The spherical transform is a topological isomorphism between the spaces $\mathcal{S}_{\mathrm{rad}}(H_n)$ and $\mathcal{S}(\Sigma)$.*

On the other hand, Lemma 3.1 implies that when f is compactly supported we can regard $\mathcal{G}f$ as a function on $\mathbb{C}^2$.

Proposition 4.2. *If f is in $\mathcal{D}_{\mathrm{rad}}(H_n)$ then $\mathcal{G}f$ extends to the holomorphic function F on $\mathbb{C}^2$ given by the rule*

$$F(\xi,\lambda) = \langle f, \check{\Phi}_{\xi,\lambda}\rangle_{H_n} \qquad \forall(\xi,\lambda)\in\mathbb{C}^2.$$

4.2. **Holomorphic versus Schwartz extensions.** Given f in $\mathcal{D}_{\mathrm{rad}}(H_n)$, we have found two ways of extending its spherical transform $\mathcal{G}f$ to a smooth function on $\mathbb{R}^2$. Namely, by Theorem 4.1, there exists a Schwartz function G on $\mathbb{R}^2$ such that $G|_\Sigma = \mathcal{G}f$, and by Proposition 4.2 the function F is the holomorphic extension of $\mathcal{G}f$ to $\mathbb{C}^2$. So $G|_\Sigma = F|_\Sigma = \mathcal{G}f$.

We observe that any two entire functions on $\mathbb{C}^2$, which coincide on Σ, are everywhere equal, so F is the unique continuation of $\mathcal{G}f$ to an entire function on $\mathbb{C}^2$.

A question arises naturally: if f is in $\mathcal{D}_{\mathrm{rad}}(H_n)$, is it true that F, when restricted to real values of (ξ,λ), is a Schwartz function on $\mathbb{R}^2$?

In the rest of this subsection we show that the answer can be negative.

Let f be a function of the form

$$f(z,t) = g(z) \otimes h(t) \qquad \forall (z,t) \in H_n$$

where h is even and compactly supported and g is nonpositive and supported in $1 < |z| < 4$, equal to -1 when $2 < |z| < 3$.

Let $\mathcal{F}h$ be the euclidean Fourier transform of h. We now show that the function $\lambda \geq 0 \mapsto |\mathcal{F}h(\lambda)|\, e^{\lambda/2}$ is not bounded. Indeed, since h is even, if it were bounded, then for every b, $0 \leq b < 1/2$, the function $\lambda \mapsto e^{b|\lambda|}\mathcal{F}h(\lambda)$ would be in $L^2(\mathbb{R})$. By the Paley–Wiener Theorem for the euclidean transform $h = \mathcal{F}^2 h$ would continue analytically to $\{w : |\mathrm{Im}\,(w)| < 1/2\}$, but this cannot be true since h has compact support. Therefore the function $\lambda \geq 0 \mapsto |\mathcal{F}h(\lambda)|\, e^{\lambda/2}$ is not bounded.

Now, if $(\xi,\lambda) \in \mathbb{R}^2 \mapsto F(\xi,\lambda)$ were rapidly decreasing, then the same would hold true for the function $\lambda \mapsto F((n+1)\lambda,\lambda)$. Note that when $\lambda > 0$

$$F((n+1)\lambda,\lambda) = \mathcal{F}h(\lambda) \int_{1<|z|<4} g(z)\, e^{-\lambda|z|^2/4}\, {}_1F_1(-\tfrac{1}{2}, n, \lambda|z|^2/2)\, dz = \mathcal{F}h(\lambda)\ G(\lambda).$$

Moreover ${}_1F_1(-\frac{1}{2}, n, x) \leq 0$ when $x \geq 2n$ and by the estimate (see [13, p. 27, formula (3)])

$$ {}_1F_1(a, n; x) = \frac{\Gamma(n)}{\Gamma(a)} e^x \, x^{a-n}(1 + O(x^{-1})), \qquad \mathrm{Re}\,(x) \to \infty, \quad a \neq 0, -1, -2, \ldots, $$

when $\lambda \to +\infty$ we obtain

$$ \begin{aligned} |F((n+1)\lambda, \lambda)| &= |\mathcal{F}h(\lambda)| \int_{1<|z|<4} g(z) \, e^{-\lambda|z|^2/4} \, {}_1F_1(-\tfrac{1}{2}, n, \lambda|z|^2/2) \, dz \\ &\geq C \, |\mathcal{F}h(\lambda)| \int_{2<|z|<3} e^{-\lambda|z|^2/4} \, e^{\lambda|z|^2/2} \, (\lambda|z|^2/2)^{-1/2-n} \, dz \\ &\geq C \, |\mathcal{F}h(\lambda)| \, e^{\lambda/2}, \end{aligned} $$

so the function $\lambda \mapsto F((n+1)\lambda, \lambda)$ is not bounded.

5. The spherical transform of radial tempered distributions

As usual, we denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ the dual pairing on $\mathbb{R}^2$ and we also write

$$ \langle \varphi, \psi \rangle_{\mathbb{R}^2} = \int_{\mathbb{R}^2} \varphi(x) \, \psi(x) \, dx \qquad\qquad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^2). $$

Let $\Pi : \mathcal{S}(H_n) \longrightarrow \mathcal{S}(H_n)$ be the averaging projector defined by

$$ \Pi f = \int_{\mathrm{U}(n)} f \circ k \; dk \qquad \forall f \in \mathcal{S}(H_n). $$

Then the Schwartz space on H_n decomposes into the direct sum $\mathcal{S}(H_n) = \mathcal{S}_{\mathrm{rad}}(H_n) \oplus \ker \Pi$ so that $\mathcal{S}_{\mathrm{rad}}(H_n)$ is isomorphic to the quotient space $\mathcal{S}(H_n)/\ker \Pi$. It follows that the dual space $\big(\mathcal{S}_{\mathrm{rad}}(H_n)\big)'$ is isomorphic to the subspace $\mathcal{S}'_{\mathrm{rad}}(H_n)$ of $\mathcal{S}'(H_n)$ consisting of all tempered distributions Λ on H_n such that

$$ \langle \Lambda, f \rangle_{H_n} = 0 \qquad \forall \, f \in \ker \Pi. $$

On the other hand the dual space of $\mathcal{S}(\Sigma)$ is naturally isomorphic to the subspace $\mathcal{S}_0'(\Sigma)$ of $\mathcal{S}'(\mathbb{R}^2)$ consisting of all tempered distributions U on $\mathbb{R}^2$ such that

$$\langle U, g\rangle_{\mathbb{R}^2} = 0 \ , \qquad \forall\, g \in \mathcal{S}(\mathbb{R}^2) \text{ such that } g = 0 \text{ on } \Sigma \ .$$

We note that the Plancherel formula can be written as

$$\langle f, \overline{g}\rangle_{H_n} = \langle \mathcal{G}f\,\mu, \overline{\mathcal{G}g}\rangle_{\mathbb{R}^2} = \langle \mathcal{G}f\,\mu, \mathcal{G}\bar{\check{g}}\rangle_{\mathbb{R}^2} \qquad f, g \in \mathcal{S}_{\text{rad}}(H_n).$$

Therefore we are led to define the spherical transform of a radial tempered distribution Λ on H_n as the distribution $\mathcal{G}\Lambda$ in $\mathcal{S}'(\mathbb{R}^2)$ given by

$$\langle \mathcal{G}\Lambda, \varphi\rangle_{\mathbb{R}^2} = \langle \Lambda, (\mathcal{G}^{-1}\varphi_{|\Sigma})\check{}\,\rangle_{H_n} \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^2). \tag{5.1}$$

Clearly, if Λ is in $\mathcal{S}_{\text{rad}}'(H_n)$, then for every function φ in $\mathcal{S}(\mathbb{R}^2)$ such that $\varphi = 0$ on Σ we have $\langle \mathcal{G}\Lambda, \varphi\rangle_{\mathbb{R}^2} = \langle \Lambda, (\mathcal{G}^{-1}\varphi_{|\Sigma})\check{}\,\rangle_{H_n} = 0$, i.e., $\mathcal{G}\Lambda$ is in $\mathcal{S}_0'(\Sigma)$.

We recall that we have denoted by m the Lebesgue measure on H_n. If f is a radial function in $L^1 \cap L^2(H_n)$, then fm is in $\mathcal{S}_{\text{rad}}'(H_n)$ and $\mathcal{G}(fm) = (\mathcal{G}f)\,\mu$ where $\mathcal{G}f$ has been defined in Subsection 4.1, so formula (5.1) provides an extension of the usual spherical transform.

Moreover it is easy to verify that $\mathcal{G}(L^j\Lambda) = \xi^j\,\mathcal{G}\Lambda$ for every Λ in $\mathcal{S}_{\text{rad}}'(H_n)$.

With this notation Theorem 4.1 extends to radial tempered distributions in the following form.

Corollary 5.1. *The spherical transform $\mathcal{G}$ is a topological isomorphism between the spaces $\mathcal{S}_{\text{rad}}'(H_n)$ and $\mathcal{S}_0'(\Sigma)$.*

We now study the behavior of the spherical transform of radial compactly supported distributions.

Proposition 5.2. *Let Λ be a radial compactly supported distribution on H_n. Then*

$$\widehat{\Lambda}\ :\ (\xi,\lambda)\longmapsto \langle \Lambda, \check{\Phi}_{\xi,\lambda}\rangle_{H_n} \tag{5.2}$$

is a holomorphic function on $\mathbb{C}^2$. Moreover $\widehat{\Lambda}\,\mu$ is in $\mathcal{S}_0'(\Sigma)$ and

$$\mathcal{G}\Lambda = \widehat{\Lambda}\,\mu\ ,$$

i.e., $\mathcal{G}\Lambda$ coincides with the function $\widehat{\Lambda}$.

Proof. Using Lemma 3.1 it is easy to prove that $\widehat{\Lambda}$ is entire. For the second part, we first check that for every ψ in $\mathcal{S}(\mathbb{R}^2)$, the integral $\int_\Sigma \widehat{\Lambda}\,\psi\,d\mu$ is absolutely convergent, and therefore $\widehat{\Lambda}\,\mu$ is in $\mathcal{S}_0'(\Sigma)$. Indeed, if $(\xi,\lambda)=\big(|\lambda|(2j+n),\lambda\big)$ is in Σ^*,

$$|\widehat{\Lambda}(\xi,\lambda)| = |\langle\Lambda,\check{\Phi}_{\xi,\lambda}\rangle_{H_n}| \le C\,\big\|\check{\Phi}_{\xi,\lambda}\big\|_{C^m(K)}\ ,$$

where $K=\operatorname{supp}\Lambda$ and for some m in $\mathbb{N}$. By Lemma 3.2 $\widehat{\Lambda}$ is slowly growing on Σ and so for every ψ in $\mathcal{S}(\mathbb{R}^2)$, the integral $\int_\Sigma \widehat{\Lambda}\,\psi\,d\mu$ is absolutely convergent.

When g is in $\mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned}
\langle \widehat{\Lambda}\,\mu,\psi\rangle_{\mathbb{R}^2} &= \int_\Sigma \widehat{\Lambda}(\xi,\lambda)\,\psi(\xi,\lambda)\,d\mu\\
&= \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}}\sum_{j=0}^{\infty}\widehat{\Lambda}\big(|\lambda|(2j+n),\lambda\big)\,\psi\big(|\lambda|(2j+n),\lambda\big)\,|\lambda|^n\,d\lambda\\
&= \lim_{N\to\infty}\frac{1}{(2\pi)^{n+1}}\int_{-N}^{N}\sum_{j=0}^{N}\widehat{\Lambda}\big(|\lambda|(2j+n),\lambda\big)\,\psi\big(|\lambda|(2j+n),\lambda\big)\,|\lambda|^n\,d\lambda\\
&= \lim_{N\to\infty}\frac{1}{(2\pi)^{n+1}}\int_{-N}^{N}\sum_{j=0}^{N}\langle\Lambda,\check{\Phi}_{|\lambda|(2j+n),\lambda}\rangle_{H_n}\,\psi\big(|\lambda|(2j+n),\lambda\big)\,|\lambda|^n\,d\lambda\\
&= \lim_{N\to\infty}\frac{1}{(2\pi)^{n+1}}\,\left\langle\Lambda,\int_{-N}^{N}\sum_{j=0}^{N}\,\check{\Phi}_{|\lambda|(2j+n),\lambda}\ \psi\big(|\lambda|(2j+n),\lambda\big)\,|\lambda|^n\,d\lambda\right\rangle_{H_n}.
\end{aligned}$$

Since

$$\lim_{N\to\infty}\frac{1}{(2\pi)^{n+1}}\int_{-N}^{N}\sum_{j=0}^{N}\check{\Phi}_{|\lambda|(2j+n),\lambda}\,\psi\big(|\lambda|(2j+n),\lambda\big)\,|\lambda|^n\,d\lambda=(\mathcal{G}^{-1}\psi_{|\Sigma})\check{}$$

uniformly on compacta and the same holds for all derivatives,

$$\langle\widehat{\Lambda}\,\mu,\psi\rangle_{\mathbb{R}^2}=\langle\Lambda,(\mathcal{G}^{-1}\psi_{|\Sigma})\check{}\,\rangle_{H_n}=\langle\mathcal{G}\Lambda,\psi\rangle_{\mathbb{R}^2}\ .\qquad\square$$

6. The operators $M_\pm$

Denote by $M_\pm$ the operators acting on a smooth function ψ on $\mathbb{R}^2$ by the rule [8]

$$\begin{aligned}M_\pm\psi(\xi,\lambda)&=\partial_\lambda\psi(\xi,\lambda)\mp n\partial_\xi\psi(\xi,\lambda)-2(n\lambda\pm\xi)\int_0^1\partial_\xi^2\psi(\xi\pm 2\lambda t,\lambda)(1-t)\,dt.\\&=\frac{1}{\lambda}\,(\lambda\partial_\lambda+\xi\partial_\xi)\,\psi(\xi,\lambda)-\frac{n\lambda\pm\xi}{2\lambda^2}\big(\psi(\xi\pm 2\lambda,\lambda)-\psi(\xi,\lambda)\big).\end{aligned}$$

Since $\lambda\partial_\lambda+\xi\partial_\xi$ is the derivative in the radial direction, the operators $M_\pm$ depend only on the restriction to the Heisenberg fan.

The operators $M_\pm$ have the following relevant property. If f is radial and f and $\mathcal{A}f$ are integrable functions on H_n then (see [8])

$$\mathcal{G}(\mathcal{A}f)=M_+(\mathcal{G}f)\qquad\text{and}\qquad\mathcal{G}(\bar{\mathcal{A}}f)=-M_-(\mathcal{G}f).\tag{6.1}$$

One can verify that

$$\langle M_+(\mathcal{G}f)\mu,\mathcal{G}h\rangle_{\mathbb{R}^2}=-\langle(\mathcal{G}f)\mu,M_-\mathcal{G}h\rangle_{\mathbb{R}^2}\qquad\forall f,h\in\mathcal{S}_{\rm rad}(H_n).$$

Hence by Theorem 4.1

$$\int_\Sigma(M_+\varphi)\,\psi\,d\mu=-\int_\Sigma\varphi\,(M_-\psi)\,d\mu\qquad\forall\varphi,\psi\in\mathcal{S}(\mathbb{R}^2).\tag{6.2}$$

According to (6.2), when U is in $\mathcal{S}_0'(\Sigma)$ we define the distribution M_+U by

$$\langle M_+U,\psi\rangle_{\mathbb{R}^2}=-\langle U,M_-\psi\rangle_{\mathbb{R}^2}\qquad\forall\psi\in\mathcal{S}(\mathbb{R}^2)$$

and similarly we define $M_- U$ with M_+ and M_- interchanged.

Clearly if $\psi_{|\Sigma} = 0$ then $(M_-\psi)_{|\Sigma} = 0$, so that M_+U is in $\mathcal{S}_0'(\Sigma)$.

Moreover it is easy to verify that (6.1) extends to distributions, i.e.

$$\mathcal{G}(\mathcal{A}\Lambda) = M_+(\mathcal{G}\Lambda) \qquad \text{and} \qquad \mathcal{G}(\bar{\mathcal{A}}\Lambda) = -M_-(\mathcal{G}\Lambda) \qquad \forall \Lambda \in \mathcal{S}'(H_n). \tag{6.3}$$

Finally, given a distribution in $\mathcal{S}_0'(\Sigma)$ of the form $F\mu$, where F is smooth and slowly growing on $\mathbb{R}^2$, we note that for all g in $\mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned}\langle M_+(F\mu), \psi\rangle_{\mathbb{R}^2} &= -\langle F\mu, M_-\psi\rangle_{\mathbb{R}^2} \\ &= -\int_\Sigma F\, M_-\psi\, d\mu \\ &= \int_\Sigma M_+F\, \psi\, d\mu = \langle (M_+F)\,\mu, \psi\rangle_{\mathbb{R}^2},\end{aligned}$$

therefore

$$M_+(F\mu) = (M_+F)\,\mu\ . \tag{6.4}$$

For later use, we prove the following estimate.

Lemma 6.1. *Let a be a positive integer, then for every ψ in $\mathcal{D}(\mathbb{R}^2)$ with support in the set $\{(\xi,\lambda) \in \mathbb{R}^2 \ :\ |\xi| \le \rho\}$*

$$\|M_\pm^a \psi\|_{L^1(\Sigma)} \le C_a\,(1+\rho)^{a+n+2} \sum_{s,r=0}^{2a} \|\partial_\lambda^s \partial_\xi^r \psi\|_{L^\infty(\mathbb{R}^2)}.$$

Proof. It is enough to prove the statement for M_+, since $M_-\check{\psi} = -\,[M_+\psi]\,\check{}$, where $\check{\psi}(\xi,\lambda) = \psi(\xi,-\lambda)$. Let W denote the operator acting on a smooth function ψ on $\mathbb{R}^2$ by

$$W\psi(\xi,\lambda) = 2\int_0^1 \partial_\xi^2 \psi(\xi + 2\lambda t, \lambda)(1-t)\, dt.$$

For every $j \ge 0$ let η_j be the function and let V_j be the operator defined by

$$\eta_j(\xi,\lambda) = \xi + (2j+n)\lambda \qquad V_j = \partial_\lambda - (2j+n)\partial_\xi.$$

With this notation $M_+ = V_0 - \eta_0 W$. Moreover, as proved in [6, Lemma 4.5], for every positive integer a,

$$M_+^a = V_0^a + \sum_{k=1}^{a} \eta_0 \cdots \eta_{k-1}\, D_{k,a}, \tag{6.5}$$

where $D_{k,a}$ is a polynomial in $V_0, \ldots, V_k, W$ of degree a such that in each monomial the operator W appears k times.

Let ψ be in $\mathcal{D}(\mathbb{R}^2)$ with support in the set $\{(\xi,\lambda) \in \mathbb{R}^2 \ :\ |\xi| \le \rho\}$. Then it is easy to see that $\operatorname{supp} D_{k,a}\psi \subseteq \{(\xi,\lambda) \in \mathbb{R}^2 \ :\ |\xi| \le c\,\rho\}$, with c depending on a. Therefore, using (4.1),

$$\begin{aligned}
\|M_+^a\psi\|_{L^1(\Sigma)} &\le \|V_0^a\psi\|_{L^1(\Sigma)} + \sum_{k=1}^{a} \|\eta_0 \cdots \eta_{k-1}\, D_{k,a}\psi\|_{L^1(\Sigma)} \\
&\le C_a\,(1+\rho)^{a+n+2} \left(\sum_{r+s\le a} \|\partial_\lambda^s \partial_\xi^r \psi\|_{L^\infty(\mathbb{R}^2)} + \sum_{k=1}^{a} \|D_{k,a}\psi\|_{L^\infty(\mathbb{R}^2)} \right).
\end{aligned}$$

We complete the proof by showing that

$$\|D_{k,a}\psi\|_{L^\infty(\mathbb{R}^2)} \le C_a \sum_{s+r\le 2a} \|\partial_\lambda^s \partial_\xi^r \psi\|_{L^\infty(\mathbb{R}^2)} \qquad k = 1, 2, \ldots, a,$$

by induction on a. Indeed, the case $a = 1$ is trivial since

$$\|W\psi\|_{L^\infty(\mathbb{R}^2)} \le 2\int_0^1 \|\partial_\xi^2\psi\|_{L^\infty(\mathbb{R}^2)}(1-t)\,dt \le C\,\|\partial_\xi^2\psi\|_{L^\infty(\mathbb{R}^2)}.$$

If $a > 1$ then either $D_{k,a} = D_{k-1,a-1}\,W$ or $D_{k,a} = D_{k,a-1}\,V_j$, for some j and $k \le a-1$. The second case is trivial. If $D_{k,a} = D_{k-1,a-1}\,W$, we note that by induction on s it is easy to verify that

$$\partial_\lambda^s \partial_\xi^r W\psi = 2\sum_{k=0}^{s} \binom{s}{k} \int_0^1 \partial_\lambda^{s-k} \partial_\xi^{r+2+k}\, \psi(\xi + 2\lambda t, \lambda)\,(2t)^k\,(1-t)\,dt.$$

Therefore

$$\|D_{k-1,a-1}\,W\psi\|_{L^\infty(\mathbb{R}^2)} \le C_a \sum_{s+r\le 2a-2} \|\partial_\lambda^s \partial_\xi^r W\psi\|_{L^\infty(\mathbb{R}^2)}$$

$$\leq C_a \sum_{s+r\leq 2a-2} \sum_{k=0}^{s} \|\partial_\lambda^{s-k}\partial_\xi^{r+2+k}\psi\|_{L^\infty(\mathbb{R}^2)}$$

$$\leq C_a \sum_{s+r\leq 2a} \|\partial_\lambda^{s}\partial_\xi^{r}\psi\|_{L^\infty(\mathbb{R}^2)}. \qquad \square$$

7. Real Paley–Wiener results for the spherical transform

Suppose that Λ is a radial tempered distribution on H_n. Motivated by [1], we define $R(\Lambda)$ in $[0,\infty]$ by

$$R(\Lambda) = \sup\{|x| \,:\, x \in \operatorname{supp}\Lambda\}$$

and we call $R(\Lambda)$ the radius of the support of the distribution Λ.

The purpose of this section is to prove real Paley–Wiener Theorems for the spherical transform; we start with a characterization of compactly supported radial distributions and then we specialize these results to square integrable radial functions and Schwartz radial functions. When a distribution U on $\mathbb{R}^2$ is of the form $U = F_U\,\mu$ with F_U a (smooth) slowly growing function on $\mathbb{R}^2$, by abuse of notation we shall also denote by U the associated function F_U.

Our first characterization reads as follows.

Theorem 7.1. *Let Λ be in $\mathcal{S}'_{\mathrm{rad}}(H_n)$. The following conditions are equivalent.*

(1) *$R(\Lambda)$ is finite;*

(2) *$\mathcal{G}\Lambda$ coincides with a smooth slowly growing function on $\mathbb{R}^2$ and for every p in $[1,\infty]$ there exists $\beta > 0$ such that*

$$\limsup_{j\to\infty} \|(1+\xi)^{-\beta}\, M_+^j \mathcal{G}\Lambda\|_{L^p(\Sigma)}^{1/j} < \infty;$$

(3) *for every large j the distribution $M_+^j\mathcal{G}\Lambda$ coincides with a smooth slowly growing function on $\mathbb{R}^2$ and there exist $\beta > 0$ and p in $[1,\infty]$ such that*

$$\liminf_{j\to\infty} \|(1+\xi)^{-\beta}\, M_+^j\mathcal{G}\Lambda\|_{L^p(\Sigma)}^{1/j} < \infty.$$

Moreover, if any of these conditions is satisfied, then for every p in $[1,\infty]$ there exists $\beta > 0$ such that

$$\lim_{j\to\infty} \|(1+\xi)^{-\beta}\, M_+^j\mathcal{G}\Lambda\|_{L^p(\Sigma)}^{1/j} = R(\Lambda)^2. \tag{7.1}$$

Since $M_-\psi = -\left[M_+\check{\psi}\right]\check{}$, where $\check{\psi}(\xi,\lambda) = \psi(\xi,-\lambda)$, we have also a corresponding analogue with M_- in place of M_+.

The proof of Theorem 7.1 is given after some preliminary results, the first of which is the following technical lemma.

Lemma 7.2. *Let $R > 0$ and j be a positive integer. Suppose that f is a smooth function on H_n with compact support in the set $\{x \in H_n : |x| > R\}$ and let $f_j = \bar{\mathcal{A}}^{-j} f$.*
Then for every N in $\mathbb{N}$

$$\|(1+\xi)^N\, \mathcal{G} f_j\|_{L^\infty(\Sigma)} \le C_N\, j^{2N}\, R^{-2j} \max_{h\le N} \sum_{\deg I+\deg J=2h} \|\bar{\mathcal{A}}^{-\deg I}\, D^J f\|_{L^1(H_n)}$$

Proof. Note that since f is supported away from the origin, the function $f_j = \bar{\mathcal{A}}^{-j} f$ is again smooth and compactly supported. Moreover,

$$\begin{aligned}\|(1+\xi)^N\, \mathcal{G} f_j\|_{L^\infty(\Sigma)} &= \|\mathcal{G}\big((I+L)^N f_j\big)\|_{L^\infty(\Sigma)}\\ &\le \|(I+L)^N f_j\|_{L^1(H_n)}.\end{aligned}$$

Clearly $(I+L)^N f_j = \sum \binom{M}{h} L^h f_j$ and by the Leibniz rule

$$\begin{aligned}
\|(I+L)^N f_j\|_{L^1(H_n)} &\leq C_N \max_{h\leq N} \|L^h f_j\|_{L^1(H_n)} \\
&\leq C_N \max_{h\leq N} \sum_{\deg I+\deg J=2h} \|(D^I \bar{\mathcal{A}}^{-j})\,(D^J f)\|_{L^1(H_n)} \\
&\leq C_N \max_{h\leq N} \sum_{\deg I+\deg J=2h} j^{|I|} \|(\bar{\mathcal{A}}^{-j-\deg I})\,(D^J f)\|_{L^1(H_n)} \\
&\leq C_N\, j^{2N}\, R^{-2j} \max_{h\leq N} \sum_{\deg I+\deg J=2h} \|\bar{\mathcal{A}}^{-\deg I}\, D^J f\|_{L^1(H_n)}. \qquad \square
\end{aligned}$$

Now we note that the spherical transform of radial compactly supported distributions satisfies a pointwise estimate on the Heisenberg fan Σ.

Proposition 7.3. *Let Λ be a radial compactly supported distribution of order N on H_n. Then for every $R > R(\Lambda)$ there exists a constant $C = C_R > 0$ such that for every j in $\mathbb{N}$*

$$|M_+^j \widehat{\Lambda}(\xi,\lambda)| \leq C\, R^{2j}\,(1+\xi)^{N/2} \qquad \forall(\xi,\lambda)\in\Sigma. \tag{7.2}$$

Proof. We have already proved in Proposition 5.2 that $\mathcal{G}\Lambda = \widehat{\Lambda}\,\mu$ and that $\widehat{\Lambda}$ extends to an entire function, so $\widehat{\Lambda}$ is in $C^\infty(\Sigma)$. Moreover by equations (6.3) and (6.4)

$$(M_+^j \widehat{\Lambda})\,\mu = M_+^j(\widehat{\Lambda}\,\mu) = M_+^j \mathcal{G}\Lambda = \mathcal{G}(\mathcal{A}^j\Lambda) = \widehat{\mathcal{A}^j\Lambda}\,\mu,$$

therefore $M_+^j\widehat{\Lambda} = \widehat{\mathcal{A}^j\Lambda}$.

Let $R > R(\Lambda)$ and choose R_1 such that $R > R_1 > R(\Lambda)$. Suppose that g is a radial test function on H_n such that $g(x) = 1$ when x is in the support of Λ and $g(x) = 0$ if $|x| > R_1$. Then for all (ξ,λ) in Σ^*,

$$\begin{aligned}
|M_+^j\widehat{\Lambda}(\xi,\lambda)| &= |\widehat{\mathcal{A}^j\Lambda}(\xi,\lambda)| = |\widehat{g\,\mathcal{A}^j\Lambda}(\xi,\lambda)| \\
&= |\langle g\,\mathcal{A}^j\Lambda, \check{\Phi}_{\xi,\lambda}\rangle_{H_n}|
\end{aligned}$$

$$
\begin{aligned}
&= |\langle \Lambda, g\, \mathcal{A}^j \Phi_{\xi,-\lambda} \rangle_{H_n}| \\
&\leq C \sum_{\deg I \leq N} \| D^I (g\, \mathcal{A}^j \Phi_{\xi,-\lambda}) \|_{L^\infty(H_n)}.
\end{aligned}
$$

We conclude by the Leibniz rule and Lemma 3.2 that

$$
|M_+^j \widehat{\Lambda}(\xi,\lambda)| \leq C\,(1+j)^N\, R_1^{2j}\,(1+\xi)^{N/2} \qquad \forall (\xi,\lambda) \in \Sigma^*,
$$

which, since $R > R_1$ and by the smoothness of $\widehat{\Lambda}$, implies (7.2). □

Conversely, it is easy to deduce that a radial tempered distribution is compactly supported when a certain limit is finite.

Proposition 7.4. *Let Λ be in $\mathcal{S}'_{\rm rad}(H_n)$. Suppose that there exists J in $\mathbb{N}$ such that for every $j \geq J$ the distribution $M_+^j \mathcal{G}\Lambda$ is of the form $G_j\,\mu$, where G_j is a locally integrable function with respect to μ. Then for every N in $\mathbb{N}$ and every p in $[1,\infty]$*

$$
\liminf_{j\to\infty} \|\,(1+\xi)^{-N}\, G_j\,\|_{L^p(\Sigma)}^{1/j} \geq R(\Lambda)^2.
$$

Proof. Suppose that $R(\Lambda) > 0$ and let $0 < \varepsilon < R(\Lambda)/2$. Then we may find a smooth function f with compact support in the set

$$
\{x \in H_n \,:\, R(\Lambda) - \varepsilon < |x| < R(\Lambda) + \varepsilon\}
$$

such that $\langle \Lambda, \check{f} \rangle_{H_n} \neq 0$. As in the previous lemma, the function f is supported away from the origin and we let $f_j = \bar{\mathcal{A}}^{-j} f$. By (5.1) and (6.3)

$$
\begin{aligned}
|\langle \Lambda, \check{f} \rangle_{H_n}| &= |\langle \Lambda, \mathcal{A}^j\, \mathcal{A}^{-j}\, \check{f} \rangle_{H_n}| = |\langle \Lambda, \mathcal{A}^j\, \check{f}_j \rangle_{H_n}| \\
&= |\langle \mathcal{A}^j\, \Lambda, \check{f}_j \rangle_{H_n}| = |\langle \mathcal{G}(\mathcal{A}^j\, \Lambda), \mathcal{G} f_j \rangle_{\mathbb{R}^2}| \\
&= |\langle M_+^j \mathcal{G}\Lambda, \mathcal{G} f_j \rangle_{\mathbb{R}^2}| = |\langle G_j\, \mu, \mathcal{G} f_j \rangle_{\mathbb{R}^2}|
\end{aligned}
$$

$$\leq \|(1+\xi)^{-N}\, G_j\|_{L^p(\Sigma)}\, \|(1+\xi)^N\, \mathcal{G} f_j\|_{L^{p'}(\Sigma)}.$$

In the case where $\|(1+\xi)^{-N}\, G_j\|_{L^p(\Sigma)} = \infty$ for all j, there is nothing to prove. Otherwise, since $|\langle \Lambda, \check{f}\rangle_{H_n}| \neq 0$,

$$\liminf_{j\to\infty} \|(1+\xi)^{-N}\, G_j\|_{L^p(\Sigma)}^{1/j} \geq \liminf_{j\to\infty} \left(\frac{|\langle \Lambda, \check{f}\rangle_{H_n}|}{\|(1+\xi)^N\, \mathcal{G} f_j\|_{L^{p'}(\Sigma)}} \right)^{1/j} = \liminf_{j\to\infty} \|(1+\xi)^N\, \mathcal{G} f_j\|_{L^{p'}(\Sigma)}^{-1/j}.$$

Since there exists M in $\mathbb{N}$ such that $\xi \mapsto (1+\xi)^{N-M}$ is in $L^{p'}(\Sigma)$, by Lemma 7.2 we conclude that

$$\begin{aligned} \|(1+\xi)^N\, \mathcal{G} f_j\|_{L^{p'}(\Sigma)} &\leq \|(1+\xi)^{N-M}\|_{L^{p'}(\Sigma)}\, \|(1+\xi)^M\, \mathcal{G} f_j\|_{L^\infty(\Sigma)} \\ &\leq C\; j^{2M}\, (R(\Lambda)-\varepsilon)^{-2j}\ , \end{aligned}$$

and the thesis follows easily.

When $R(\Lambda) = \infty$ we use the same arguments to show that $\liminf_{j\to\infty} \|(1+\xi)^{-N}\, G_j\|_{L^p(\Sigma)}^{1/j} \geq R$ for every $R > 0$. □

Putting together Proposition 7.3 and Proposition 7.4, we obtain the following criterion, by which we can measure the size of the support of a radial compactly supported distribution.

Corollary 7.5. *Let Λ be a radial compactly supported distribution of order N. Then*

$$\lim_{j\to\infty} \|(1+\xi)^{-N/2}\, M_+^j \widehat{\Lambda}\|_{L^\infty(\Sigma)}^{1/j} = R(\Lambda)^2.$$

Proof. From the pointwise estimate (7.2), we deduce that for every $R > R(\Lambda)$

$$\limsup_{j\to\infty} \|(1+\xi)^{-N/2}\, M_+^j \widehat{\Lambda}\|_{L^\infty(\Sigma)}^{1/j} \leq R^2,$$

therefore $\limsup_{j\to\infty} \|(1+\xi)^{-N/2}\, M_+^j \widehat{\Lambda}\|_{L^\infty(\Sigma)}^{1/j} \leq R(\Lambda)^2$. The thesis follows by Proposition 7.4. □

Proof of Theorem 7.1. If Λ is compactly supported and of order N then by Proposition 5.2 it coincides with a smooth slowly growing function G on $\mathbb{R}^2$. If $\beta > 0$ is such that $(1+\xi)^{N/2-\beta}$ is in $L^p(\Sigma)$, we have

$$\|(1+\xi)^{-\beta}\, M_+^j G\|_{L^p(\Sigma)} \le \|(1+\xi)^{N/2-\beta}\|_{L^p(\Sigma)} \|(1+\xi)^{-N/2}\, M_+^j G\|_{L^\infty(\Sigma)}. \tag{7.3}$$

Hence by Corollary 7.5 we have that (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is trivial and the implication (3) $\Rightarrow$ (1) is a consequence of Proposition 7.4. Finally (7.1) follows by (7.3), Proposition 7.4 and Corollary 7.5. □

7.1. **Square-integrable functions.**

Theorem 7.6. *Suppose that for every $j \ge 0$ the function $M_+^j \psi$ is in $L^2(\Sigma)$. Then the function f such that $\mathcal{G} f = \psi$ is square integrable and*

$$\lim_{j\to\infty} \|\, M_+^j \psi\, \|_{L^2(\Sigma)}^{1/j} = R(f)^2.$$

Proof. By Proposition 7.4 it is enough to check that $\limsup_{j\to\infty} \|M_+^j \psi\|_{L^2(\Sigma)}^{1/j} \le R(f)^2$, and this is easily established by using the Plancherel formula. Indeed, when $R(f)$ is finite,

$$\|M_+^j \psi\, \|_{L^2(\Sigma)} = \|\mathcal{A}^j\, f\|_{L^2(H_n)} \le R(f)^{2j}\, \|f\|_{L^2(H_n)}.$$ □

Corollary 7.7. *$\mathcal{G}$ is a bijection from the space $L^2_{\mathrm{rad},R}(H_n)$ of square integrable radial functions f such that $R(f) \le R$ onto $\{\psi \in L^2(\Sigma)\ : \lim_{j\to\infty} \|\, M_+^j \psi\, \|_{L^2(\Sigma)}^{1/j} \le R^2\}$.*

7.2. **Schwartz functions.**

The purpose of this subsection is to prove the following characterization.

Theorem 7.8. *Let f be in $\mathcal{S}_{\mathrm{rad}}(H_n)$. The following conditions are equivalent.*

(1) *$R(f)$ is finite;*

(2) *for every* $h \geq 0$ *and every* p *in* $[1, \infty]$, $\limsup_{j\to\infty} \| \, \xi^h \, M_+^j \mathcal{G} f \, \|_{L^p(\Sigma)}^{1/j}$ *is finite;*

(3) *there exists* p *in* $[1, \infty]$ *such that* $\liminf_{j\to\infty} \| \, M_+^j \mathcal{G} f \, \|_{L^p(\Sigma)}^{1/j}$ *is finite.*

Moreover, if any of these conditions is satisfied, then for every $h \geq 0$ *and every* p *in* $[1, \infty]$,

$$\lim_{j\to\infty} \| \, (1+\xi)^h \, M_+^j \mathcal{G} f \, \|_{L^p(\Sigma)}^{1/j} = R(f)^2.$$

Note that the implication (2) $\Rightarrow$ (3) is trivial, and that (3) $\Rightarrow$ (1) follows from Proposition 7.4. In the next proposition we prove the implication (1) $\Rightarrow$ (2).

Proposition 7.9. *Suppose that* f *is a radial Schwartz function on* H_n. *Then for every* $h \geq 0$ *and every* p *in* $[1, \infty]$

$$\limsup_{j\to\infty} \| \, (1+\xi)^h \, M_+^j \mathcal{G} f \, \|_{L^p(\Sigma)}^{1/j} \leq R(f)^2.$$

Proof. If $R(f) = \infty$ there is nothing to prove. If $R(f) = 0$, then $f = 0$ and the conclusion is again trivial. We therefore suppose that $R(f)$ is positive. Note that

$$\xi^h \, M_+^j \mathcal{G} f(\lambda, \xi) = \mathcal{G}\big(L^h \mathcal{A}^j f\big)$$

and when $j \geq 2h$, by the Leibniz rule

$$\begin{aligned} |L^h \mathcal{A}^j \, f| &= \Bigg| \sum_{\deg I + \deg J = 2h} c_{h,I,J} \, (D^I \mathcal{A}^j) \, (D^J f) \Bigg| \\ &\leq \sum_{\deg I + \deg J = 2h} |c_{h,I,J}| \, j^{|I|} \, |\mathcal{A}^{j - \deg I}| \, |D^J f|. \end{aligned}$$

Therefore

$$\begin{aligned} \| \, \xi^h \, M_+^j \mathcal{G} f \, \|_{L^\infty(\Sigma)} &\leq \|L^h \mathcal{A}^j \, f\|_{L^1(H_n)} \\ &\leq C_h \, j^{2h} \sum_{q \leq 2h} \max_{\deg J = 2h - q} \|\mathcal{A}^{j-q} \, D^J f\|_{L^1(H_n)} \\ &\leq C_h \, j^{2h} \sum_{q \leq 2h} R(f)^{2j-2q} \max_{\deg J = 2h - q} \|D^J f\|_{L^1(H_n)} \end{aligned}$$

$$= C_{f,h}\, j^{2h}\, R(f)^{2j}.$$

We note that for a sufficiently big integer M the function $(\lambda, \xi) \mapsto (1+\xi)^{-M}$ is in $L^p(\Sigma)$, so that

$$\begin{aligned}\| (1+\xi)^h\, M_+^j \mathcal{G} f \,\|_{L^p(\Sigma)} &\le C\ \|(1+\xi)^{M+h}\, M_+^j \mathcal{G} f\,\|_{L^\infty(\Sigma)} \\ &\le C_{f,M,h}\, \left(1 + j^{2M+2h}\right)\, R(f)^{2j},\end{aligned}$$

and taking the j-th root, the desired inequality follows. □

Corollary 7.10. *Suppose that f is a radial Schwartz function on H_n and let $1 \le p \le \infty$. Then for every h in $\mathbb{N}$*

$$\lim_{j\to\infty} \| (1+\xi)^h\, M_+^j \mathcal{G} f\, \|_{L^p(\Sigma)}^{1/j} = R(f)^2.$$

Proof. Since $\|(1+\xi)^h\, M_+^j \mathcal{G} f\|_{L^p(\Sigma)} \ge \|\, M_+^j \mathcal{G} f\,\|_{L^p(\Sigma)}$, by Proposition 7.4 we obtain

$$\liminf_{j\to\infty} \| (1+\xi)^h\, M_+^j \mathcal{G} f\, \|_{L^p(\Sigma)}^{1/j} \ge \liminf_{j\to\infty} \|\, M_+^j \mathcal{G} f\, \|_{L^p(\Sigma)}^{1/j} \ge R(f)^2.$$

The thesis follows from Proposition 7.9. □

8. Paley–Wiener theorems for the inverse spherical transform

In this section we describe the inverse spherical transform of compactly supported distributions in $\mathcal{S}_0'(\Sigma)$.

Given a compactly supported distribution in $\mathcal{S}'(\mathbb{R}^2)$, we define the function f_U on the Heisenberg group by

$$f_U(z,t) = \langle U, \Phi_{(\cdot)}(z,t)\rangle_{\mathbb{R}^2} \qquad \forall (z,t) \in H_n.$$

An easy consequence of Lemma 3.1 is the following.

Lemma 8.1. *Let U be a compactly supported distribution in $\mathcal{S}'(\mathbb{R}^2)$. Then the function*

$$(x, y, t) \mapsto f_U(x + iy, t) = \langle U, \Phi_{(\cdot)}(x + iy, t)\rangle_{\mathbb{R}^2}$$

extends to a holomorphic function on $\mathbb{C}^{2n+1}$.

If U is in $\mathcal{S}'(\mathbb{R}^2)$, define

$$\rho(U) = \sup \{|\xi| \,:\, (\xi, \lambda) \in \operatorname{supp} U\} ,$$

so that a distribution U in $\mathcal{S}'_0(\Sigma)$ is compactly supported if and only if $\rho(U)$ is finite.

In the next proposition we prove that if U is a compactly supported distribution in $\mathcal{S}'_0(\Sigma)$ then the function f_U is a slowly growing function on H_n and it coincides with the inverse spherical transform of U.

Proposition 8.2. *Let U be a compactly supported distribution in $\mathcal{S}'_0(\Sigma)$ and let, as before,*

$$f_U(z, t) = \langle U, \Phi_{(\cdot)}(z, t)\rangle_{\mathbb{R}^2} \qquad \forall (z, t) \in H_n.$$

Then $U = \mathcal{G}(f_U\, m)$ and f_U is a slowly growing function on H_n together with all its derivatives. Moreover, for every $\rho > \rho(U)$ there exist $C = C_\rho$ and M such that for all $j \geq 0$

$$\left|L^j f_U(z, t)\right| \leq C\, (1 + j)^k\, \rho^j\, \big(1 + |(z, t)|\big)^M \qquad \forall (z, t) \in H_n, \tag{8.1}$$

where k is the order of U.

Remark 8.3. Observe that if U is a distribution in $\mathcal{S}'(\mathbb{R}^2)$ with compact support in Σ, then the function f_U may not be slowly growing. Indeed let $U = \partial_\xi \delta_{(n,1)}$ where $\delta_{(n,1)}$ is the Dirac measure at the point $(n, 1)$ in Σ. Then, reasoning as in Lemma 3.1, when (z, t) is in H_n

$$f_U(z, t) = \partial_\xi \Phi_{\xi,\lambda}\,{}_{|(n,1)}(z, t) = -\frac{e^{it}\, e^{-|z|^2/4}}{2} \sum_{k=1}^{\infty} \frac{(|z|^2/2)^k}{k\,(n)_k}.$$

Since $k < k+n$ and $(n)_k \leq (n+k-1)!$ we obtain when $|z|$ is large

$$|f_U(z,t)| > \frac{e^{-|z|^2/4}}{2} \sum_{k=1}^{\infty} \frac{(|z|^2/2)^k}{(n+k)!} \sim \frac{e^{|z|^2/4}}{2(|z|^2/2)^n}.$$

This is due, much as in Subsection 4.2, to the fact that the holomorphic extension of spherical functions does not satisfy good estimates away from the Heisenberg fan. The main point in the proof of Proposition 8.2 is that, according to formula (8.2), if U is in $\mathcal{S}'_0(\Sigma)$ one is allowed to choose a different extension.

Proof. By Theorem 5.1 there exists Λ in $\mathcal{S}'_{\mathrm{rad}}(H_n)$ such that $\mathcal{G}\Lambda = U$. Let g be in $\mathcal{D}(H_n)$, then

$$\begin{aligned}
\langle f_U, g\rangle_{H_n} &= \int_{H_n} f_U(z,t)\, g(z,t)\, dz\, dt \\
&= \int_{H_n} \langle U, \Phi_{(\cdot)}(z,t)\rangle_{\mathbb{R}^2}\; g(z,t)\, dz\, dt \\
&= \Big\langle U, \int_{H_n} \Phi_{(\cdot)}(z,t)\, g(z,t)\, dz\, dt \Big\rangle_{\mathbb{R}^2} \\
&= \langle U, \mathcal{G}\breve{g}\rangle_{\mathbb{R}^2} \\
&= \langle \Lambda, g\rangle_{H_n}\ .
\end{aligned}$$

Hence the distribution Λ coincides with the function f_U, which is smooth.

We first prove the estimate (8.1). Fix (z,t) in H_n. Let k be the order of U, let $\rho > \rho(U)$ and denote by B_ρ the ball of radius ρ in $\mathbb{R}^2$. Then for every $j \geq 0$

$$\begin{aligned}
\big|L^j f_U(z,t)\big| &= \big|\langle U, D^I \Phi_{(\cdot)}(z,t)\rangle_{\mathbb{R}^2}\big| = \big|\langle\, U, \xi^j \Phi_{(\cdot)}(z,t)\rangle_{\mathbb{R}^2}\big| \\
&\leq C \inf\Big\{ \|\xi^j \varphi^{z,t}\|_{C^k(B_\rho)} \,:\, \varphi^{z,t} \in C^k(\mathbb{R}^2), \quad {\varphi^{z,t}}_{|\Sigma\cap B_\rho} = \Phi_{(\cdot)}(z,t) \Big\} \qquad (8.2)\\
&\leq C_\rho\, (1+j)^k \rho^j \inf\Big\{ \|\varphi^{z,t}\|_{C^k(B_\rho)} \,:\, \varphi^{z,t} \in C^k(\mathbb{R}^2), \quad {\varphi^{z,t}}_{|\Sigma\cap B_\rho} = \Phi_{(\cdot)}(z,t) \Big\}
\end{aligned}$$

In order to obtain the desired estimate we shall choose a suitable extension $\varphi^{z,t}$ of $\Phi_{(\cdot)}(z,t)$.

Let ψ be a smooth function on $\mathbb{R}^2$ with compact support such that $\psi_{|B_\rho} = 1$. By Theorem 4.1 there exists u in $\mathcal{S}_{\mathrm{rad}}(H_n)$ such that $\mathcal{G}u = \psi_{|\Sigma}$. If $\nu_{(z,t)}$ denotes the measure defined by

$$\int_{H_n} g(w,s)\, d\nu_{(z,t)}(w,s) = \int_{U(n)} g(kz,t)\, dk \qquad \forall g \in C_c(H_n),$$

then for every (ξ,λ) in $B_\rho \cap \Sigma$

$$\Phi_{\xi,\lambda}(z,t) = \mathcal{G}\check{\nu}_{(z,t)}(\xi,\lambda) = \mathcal{G}\check{\nu}_{(z,t)}(\xi,\lambda)\,\psi(\xi,\lambda) = \mathcal{G}\left(\check{\nu}_{(z,t)} * u\right)(\xi,\lambda).$$

Since $\check{\nu}_{(z,t)} * u$ belongs to $\mathcal{S}_{\mathrm{rad}}(H_n)$, then by Theorem 4.1 there exist $\varphi^{z,t}$ in $\mathcal{S}(\mathbb{R}^2)$ and $M \geq 0$ such that

$$\varphi^{z,t}(\xi,\lambda) = \mathcal{G}\left(\check{\nu}_{(z,t)} * \psi\right)(\xi,\lambda) \qquad \forall (\xi,\lambda) \in \Sigma$$

and

$$\|\varphi^{z,t}\|_{C^k(B_\rho)} \leq C\, \|\check{\nu}_{(z,t)} * u\|_{(M)}.$$

Moreover

$$\varphi^{z,t}(\xi,\lambda) = \Phi_{(\xi,\lambda)}(z,t) \qquad \forall (\xi,\lambda) \in B_\rho \cap \Sigma.$$

If $\tau_{(w,s)}u(w',s') = u\big((w,s)^{-1}(w',s')\big)$ denotes the left translation, then

$$\begin{aligned}
\|\varphi^{z,t}\|_{C^k(B_\rho)} &\leq C\, \|\nu_{(z,t)} * u\|_{(M)} \\
&= C \left\| \int_{H_n} \tau_{(w,s)} u\, d\nu_{(z,t)}(w,s) \right\|_{(M)} \\
&\leq C \int_{H_n} \|\tau_{(w,s)}u\|_{(M)}\, d\nu_{(z,t)}(w,s) \\
&\leq C \int_{H_n} (1 + |w|^4 + s^2)^{M/4}\, d\nu_{(z,t)}(w,s) \\
&= C\left(1 + |(z,t)|\right)^M.
\end{aligned}$$

Therefore there exists M such that for all $j \geq 0$

$$\left|L^j f_U(z,t)\right| \leq C\,(1+j)^k\,\rho^j\,\big(1+|(z,t)|\big)^M \qquad \forall (z,t) \in H_n.$$

The proof above can be adapted to prove that for every differential operator D^I of the form (2.1) there exists $M > 0$ such that

$$|D^I f_U(z,t)| \leq C\,(1+|(z,t)|)^M \qquad \forall (z,t) \in H_n.$$

Indeed, note that

$$D^I f_U(z,t) = \langle U, D^I \Phi_{(\cdot)}(z,t)\rangle_{\mathbb{R}^2}\ ,$$

therefore

$$|D^I f_U(z,t)| \leq C \inf\left\{\|\varphi^{z,t,I}\|_{C^k(B_\rho)}\ :\ \varphi^{z,t,I} \in C^k(\mathbb{R}^2) \quad {\varphi^{z,t,I}}_{|\Sigma\cap B_\rho} = D^I \Phi_{(\cdot)}(z,t)\right\}.$$

Fix (z,t) in H_n and consider the distribution $D^I_{(z,t)}\nu_{(z,t)}$ defined by the rule

$$\langle D^I_{(z,t)}\nu_{(z,t)}, \varphi\rangle_{H_n} = D^I\left(\int_K \varphi(kz,t)\,dk\right).$$

Then $D^I_{(z,t)}\nu_{(z,t)}$ is a radial distribution supported in the orbit of (z,t), hence it has compact support. So, for ψ and u as above, $D^I_{(z,t)}\check{\nu}_{(z,t)} * u$ is in $\mathcal{S}_{\mathrm{rad}}(H_n)$ and by [4, Proposition 3.2] there exists $\varphi^{z,t,I}$ in $C^k(\mathbb{R}^2)$ and M such that

$$\varphi^{z,t,I}_{|\Sigma} = \mathcal{G}(D^I_{(z,t)}\check{\nu}_{(z,t)} * u) \qquad\qquad \|\varphi^{z,t,I}\|_{C^k(B_\rho)} \leq C\,\|D^I_{(z,t)}\check{\nu}_{(z,t)} * u\|_{(M)}$$

Since $\mathcal{G}u_{|\Sigma\cap B_\rho} = \psi_{|\Sigma\cap B_\rho} = 1$,

$$\varphi^{z,t,I}(\xi,\lambda) = \mathcal{G}(D^I_{(z,t)}\check{\nu}_{(z,t)} * u)(\xi,\lambda) = \mathcal{G}(D^I_{(z,t)}\check{\nu}_{(z,t)})(\xi,\lambda) \qquad \forall (\xi,\lambda) \in \Sigma\cap B_\rho,$$

and by Proposition 5.2

$$\mathcal{G}(D^I_{(z,t)}\check{\nu}_{(z,t)})(\xi,\lambda) = \langle D^I_{(z,t)}\check{\nu}_{(z,t)}, \check{\Phi}_{\xi,\lambda}\rangle_{\mathbb{R}^2} = D^I\left((w,s) \mapsto \int_K \Phi_{\xi,\lambda}(kw,s)\,dk\right)(z,t)$$

$$= D^I \Phi_{\xi,\lambda}(z,t).$$

Finally, reasoning as before,

$$\|\varphi^{z,t,I}\|_{C^k(B_\rho)} \leq C \, \|D^I_{(z,t)} \check{\nu}_{(z,t)} * u\|_{(M)} \leq C \left(1 + |(z,t)|\right)^M. \qquad \square$$

Our characterization of the inverse spherical transform of compactly supported distributions is the following.

Theorem 8.4. *Let U be in $\mathcal{S}_0'(\Sigma)$. The following conditions are equivalent.*

(1) *$\rho(U)$ is finite;*

(2) *$\mathcal{G}^{-1}U$ coincides with a smooth slowly growing function function on H_n and for every p in $[1,\infty]$ there exists $\beta > 0$ such that*

$$\limsup_{j\to\infty} \| \, (1+\mathcal{A})^{-\beta} \, L^j \mathcal{G}^{-1} U \, \|^{1/j}_{L^p(H_n)} < \infty;$$

(3) *for every large j the distribution $L^j\mathcal{G}^{-1}U$ coincides with a measurable function on H_n and there exist $\beta > 0$ and p in $[1,\infty]$ such that*

$$\liminf_{j\to\infty} \| \, (1+\mathcal{A})^{-\beta} \, L^j \mathcal{G}^{-1} U \, \|^{1/j}_{L^p(H_n)} < \infty.$$

Moreover, if any of these conditions is satisfied, then $\mathcal{G}^{-1}U$ is a smooth slowly growing function on H_n and for every p in $[1,\infty]$ there exists $\beta > 0$ such that

$$\lim_{j\to\infty} \| \, (1+\mathcal{A})^{-\beta} \, L^j \mathcal{G}^{-1} U \, \|^{1/j}_{L^p(H_n)} = \rho(U). \tag{8.3}$$

As in the previous section, we split the proof of our characterization into smaller parts.

Proposition 8.5. *Let U be in $\mathcal{S}_0'(\Sigma)$. Suppose that there exists J in $\mathbb{N}$ such that for every $j \geq J$ the distribution $L^j\mathcal{G}^{-1}U$ is of the form $f_j \, m$, where f_j is a locally integrable function* *on H_n. Then for every N in $\mathbb{N}$ and every p in $[1,\infty]$*

$$\liminf_{j\to\infty} \| \, (1+\mathcal{A})^{-N} \, f_j \, \|_{L^p(H_n)}^{1/j} \geq \rho(U).$$

Proof. For the reader's convenience we write the proof although it follows the lines of that of Proposition 7.4. We may suppose that $\rho(U)$ is positive, because in the case where $\rho(U)=0$, there is nothing to prove.

Let $\| \, (1+\mathcal{A})^{-N} \, f_j \, \|_{L^p(H_n)} < \infty$. Suppose that $0<\varepsilon<\rho(U)/2$ and let ψ be smooth function on $\mathbb{R}^2$ with compact support in the set

$$\{(\xi,\lambda)\in\mathbb{R}^2 \, : \, \rho(U)-\varepsilon<\xi<\rho(U)+\varepsilon\}$$

such that $\langle U,\psi\rangle_{\mathbb{R}^2}\neq 0$. For every nonnegative integer j, define a smooth function on $\mathbb{R}^2$ with compact support by $\psi_j(\xi,\lambda)=\xi^{-j}\,\psi(\xi,\lambda)$ for every (ξ,λ) in $\mathbb{R}^2$. Then for $1\leq p\leq\infty$,

$$\begin{aligned}
|\langle U,\psi\rangle_{\mathbb{R}^2}| &= |\langle \xi^j \, U,\psi_j\rangle_{\mathbb{R}^2}| = |\langle L^j\mathcal{G}^{-1}U, (\mathcal{G}^{-1}\psi_{j_{|\Sigma}})\check{}\,\rangle_{H_n}| \\
&\leq \|(1+\mathcal{A})^{-N} f_j\|_{L^p(H_n)} \, \|(1+\mathcal{A})^{N}\mathcal{G}^{-1}\psi_{j_{|\Sigma}}\|_{L^{p'}(H_n)}
\end{aligned}$$

Let a be a positive integer such that $\|(1+\mathcal{A})^{N-a}\|_{L^{p'}(H_n)}<\infty$, then by Lemma 6.1

$$\begin{aligned}
\|(1+\mathcal{A})^{N}\,\mathcal{G}^{-1}\psi_{j_{|\Sigma}}\|_{L^{p'}(H_n)} &\leq \|(1+\mathcal{A})^{N-a}\|_{L^{p'}(H_n)}\|(1+M_+)^a\,\psi_j\|_{L^1(\Sigma)} \\
&\leq C_a\,\big(\rho(U)+\varepsilon\big)^a \sum_{s,r=1}^{2a} \|\partial_\lambda^s\partial_\xi^r\psi_j\|_{L^\infty(\mathbb{R}^2)} \\
&\leq \; C_a\,\big(\rho(U)+\varepsilon\big)^a \; j^{2a}\,\big(\rho(U)-\varepsilon\big)^{-j}.
\end{aligned}$$

Therefore

$$\|(1+\mathcal{A})^{-N}\,f_j\|_{L^p(H_n)} \geq |\langle U,\psi\rangle_{\mathbb{R}^2}|\,C_{a,\varepsilon}\,j^{-2a}\,\big(\rho(U)-\varepsilon\big)^{j}$$

and the thesis follows. Similar considerations can be used in the case where $\rho(U)=\infty$. $\square$

Proposition 8.6. *Let U be in $\mathcal{S}_0'(\Sigma)$ with $\rho(U) < \infty$. Then $\mathcal{G}^{-1}U$ coincides with a smooth slowly growing function f on H_n and for every p in $[1,\infty]$ there exists $h > 0$ such that*

$$\limsup_{j\to\infty} \| \, (1+\mathcal{A})^{-h} \, L^j f \, \|_{L^p(H_n)}^{1/j} \leq \rho(U).$$

Proof. Since $\rho(U) < \infty$, the distribution U is compactly supported and therefore $\mathcal{G}^{-1}U$ coincides with the smooth function f_U on H_n by Lemma 8.1 and f_U is slowly growing by Proposition 8.2. Moreover, the estimate (8.1) holds: if $\rho > \rho(U)$ and k is the degree of U, there exists M such that for all $j \geq 0$

$$\left| L^j f(z,t) \right| \leq C \, (1+j)^k \, \rho^j \, \big(1 + |(z,t)|\big)^M.$$

Let p in $[1,\infty]$ be fixed and choose h such that $(1+\mathcal{A})^{-h+M/2}$ is in $L^p(H_n)$. Then for every $\rho > \rho(U)$

$$\begin{aligned} \| \, (1+\mathcal{A})^{-h} \, L^j f \, \|_{L^p(H_n)} &\leq \| \, (1+\mathcal{A})^{-h+M/2} \, \|_{L^p(H_n)} \, \| \, (1+\mathcal{A})^{-M/2} \, L^j f \, \|_{L^\infty(H_n)} \\ &\leq C \, (1+j)^k \, \rho^j \end{aligned}$$

so that $\limsup_{j\to\infty} \| \, (1+\mathcal{A})^{-h} \, L^j f \, \|_{L^p(H_n)}^{1/j} \leq \rho$, for every $\rho > \rho(U)$. □

In the case of Schwartz functions, we obtain the following properties. For F in $\mathcal{S}(\Sigma)$ we denote $\rho(F) = \rho(F\mu)$, so that

$$\rho(F) = \sup\{\xi \; : \; F(\xi,\lambda) \neq 0 \quad \text{and} \quad (\xi,\lambda) \in \Sigma\}$$

Proposition 8.7. *Let $1 \leq p \leq \infty$ and let F be in $\mathcal{S}(\Sigma)$. Then for every $h \geq 0$,*

$$\lim_{j\to\infty} \| \, (1+\mathcal{A})^h \, L^j \mathcal{G}^{-1} F \, \|_{L^p(H_n)}^{1/j} = \rho(F).$$

Proof. Suppose $0 < \rho(F) < \infty$. If $\gamma > 0$ is big enough so that $(1+\mathcal{A})^{-\gamma}$ is in $L^p(H_n)$ by Lemma 6.1 we obtain

$$\begin{aligned} \| \ (1+\mathcal{A})^h \, L^j \mathcal{G}^{-1} F \ \|_{L^p(H_n)} &\leq \| \ (1+\mathcal{A})^{-\gamma} \ \|_{L^p(H_n)} \| \ (1+\mathcal{A})^{h+\gamma} \ L^j \mathcal{G}^{-1} F \, \|_{L^\infty(H_n)} \\ &\leq C \, \| \ (1+M_+)^{h+\gamma} \left(\xi^j \, F \right) \|_{L^1(\Sigma)} \\ &\leq C \, j^{2h+2\gamma} \, (\rho(F))^j \, . \end{aligned}$$

Hence

$$\limsup_{j\to\infty} \| \ (1+\mathcal{A})^h \ L^j \mathcal{G}^{-1} F \ \|_{L^p(H_n)}^{1/j} \leq \rho(F).$$

and the thesis follows from Propostition 8.5. The cases $\rho(F) = 0, \infty$ are trivial. □

Theorem 8.8. *Let F be in $\mathcal{S}(\Sigma)$. The following conditions are equivalent.*

(1) *$\rho(F)$ is finite;*

(2) *for every $h \geq 0$ and every p in $[1, \infty]$, $\limsup_{j\to\infty} \| \mathcal{A}^h \, L^j \mathcal{G}^{-1} F \ \|_{L^p(H_n)}^{1/j}$ is finite;*

(3) *there exists p in $[1, \infty]$ such that $\liminf_{j\to\infty} \| \, L^j \mathcal{G}^{-1} F \ \|_{L^p(H_n)}^{1/j}$ is finite.*

Moreover, if any of these conditions is satisfied, then for every $h \geq 0$ and every p in $[1, \infty]$,

$$\lim_{j\to\infty} \| \ (1+\mathcal{A})^h \ L^j \mathcal{G}^{-1} F \ \|_{L^p(H_n)}^{1/j} = \rho(F).$$

Proof. The implication (1) ⇒ (2) follows by Propostion 8.7. The implication (2) ⇒ (3) is trivial. The implication (3) ⇒ (1) follows by Propostion 8.5. □

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