ON A NECESSARY CONDITION FOR B-SPLINE GABOR FRAMES

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Abstract. In a previous note K. Gröchenig et al. prove that if g is a continuous function with compact support such that the translates of g form a partition of unity, then g cannot generate a Gabor frame for integer values of the frequency shift parameter b greater than 1 [6]. We give a simpler proof of this result which applies also to windows g which are neither continuous nor with compact support. Our proof is based on a necessary condition for Gabor frames due to C. E. Heil and D. F. Walnut.

1. Introduction

Let g be a function in $L^2(\mathbf{R})$ and a, b two positive parameters, the family $(g, a, b) = \{g_{ma.nb},\, m, n \in \mathbf{Z}\}$ defined by

$$g_{ma,nb}(x) = g(t - na)\, e^{2\pi imbt}$$

is called *a Gabor frame* in $L^2(\mathbf{R})$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{m,n} |\langle f, g_{ma,nb}\rangle|^2 \leq B\|f\|^2 \tag{1.1}$$

for all functions $f \in L^2(\mathbf{R})$. The function g is called *window function* and the parameters a and b are called respectively *time shift* and *frequency shift* parameters. Denoting by $F_{g,a,b} : L^2(\mathbf{R}) \to l^2(\mathbf{Z}^2)$ the operator

$$F_{g,a,b} f = (\langle f, g_{ma,nb}\rangle)_{m,n},$$

and by $F^*_{g,a,b}$ its adjoint, we may rewrite condition (1.1) as $AI \leq F^*_{g,a,b}F_{g,a,b} \leq BI$. The operator ${F_{g,a,b}}^*F_{g,a,b}$ is called the *frame operator.*

We shall now review few basic facts from the theory of Gabor frames. We refer to the book of Gröchenig for a comprehensive treatment of Gabor frames [5]. If (g, a, b) is a frame, then the parameters a, b must satisfy the condition $ab < 1$. Denote by $\hat{g}$ the Fourier transform of g. The family (g, a, b) is a frame if and only if $(\hat{g}, b, a)$ is a frame. Finally, if (g, a, b) is a frame, the function $\tilde{g} = ({F_{g,a,b}}^*F_{g,a,b})^{-1}g$ also generate a frame for a and b and each element of $L^2(\mathbf{R})$ can be written as

$$f = \sum_{m,n} \langle f, \tilde{g}_{ma,nb}\rangle g_{ma,nb}, \tag{1.2}$$

where $\tilde{g}_{ma,nb}(x) = \tilde{g}(t - na)e^{2\pi imbt}$. The function $\tilde{g}$ is called *frame dual function* and the family $(\tilde{g}, a, b)$ *dual frame.*

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The literature on Gabor frames has expanded considerably in the last years. In [1] O. Christensen finds the conditions on the parameters a and b such that (g, a, b) is a frame with a dual generated by a finite linear combination of the translates of g. More recently, in [2], the same author shows that a frame $(g, 1, b)$ always has a compactly supported dual window if g is bounded, supported in $[-1, 1]$ and $b \in]1/2, 1[$. R. S. Laugesen introduces a method for constructing dual Gabor window functions that are polynomial splines [9]. We refer the reader to the recent article of K. Gröchenig [7] for a very general result on necessary and sufficient conditions for Gabor frames.
In applications, once the window function has been chosen, the first question to investigate for Gabor analysis is to find the values of the time-frequency parameters a, b such that (g, a, b) is a frame. A useful tool in this context is the A. Ron and Z. Shen criterion [10]. By using this criterion, K. Gröchenig, A. J. E. Janssen, N. Kaiblinger and G. E. Pfander have proved that (g, a, b) cannot be a frame for $a > 0$ and b integer greater than 1. In the next section we give a simpler proof which applies to more general windows. The proof is a straightforward consequence of a necessary condition of Heil and Walnut (see [8] p. 649). The end of the section is dedicated to a discussion on the B-splines.

2. Proof of the result

We recall the necessary condition in [8]

Proposition 2.1. *Let $g \in L^2(\mathbf{R})$ and $a, b > 0$ such that (g, a, b) is a frame. Then there exist two constants $0 < A \le B < \infty$ such that*

$$A \le \sum_{n \in \mathbf{Z}} |g(t - na)|^2 \le B \qquad a.e. \tag{2.1}$$

Remark 2.2. Note that in particular g must be bounded. Since also $(\hat{g}, b, a)$ is a frame, condition (2.1) holds for the function $\sum_{n \in \mathbf{Z}} |\hat{g}(t - nb)|^2$ as well.

Theorem 2.3. *Let g be a bounded function in $L^1(\mathbf{R})$ such that*

$$\sum_{k \in \mathbf{Z}} g(t - k) = 1 \qquad t \in \mathbf{R} \tag{2.2}$$

and $|\hat{g}(\xi)| \le C(1 + |\xi|)^{-s}$, for some $s > \frac{1}{2}$. Then (g, a, b) cannot be a frame for values of the parameters $a > 0$ and b integer greater than 1.

Proof. By the Poisson summation formula, the Fourier series of the periodic function $\sum_k g(t - k)$ is

$$\sum_{m \in \mathbf{Z}} \hat{g}(m) e^{2\pi i m t}.$$

Assumption (2.2) implies that $\hat{g}(m) = 0$ for all $m \neq 0$. Therefore the sum

$$S(t) = \sum_{n \in \mathbf{Z}} |\hat{g}(t - nb)|^2$$

vanishes for $t = 1$, since all summands vanish. The decay assumption on $\hat{g}$ implies that the series defining S converges uniformly. Hence S is a continuous function and $\operatorname{ess\,inf} S = 0$. Thus, by the remark, (g, a, b) cannot be a frame. $\square$

We note that B-splines satisfy the assumptions of the theorem; the cardinal B-spline of order M is defined by

$$s_M(t) = \frac{1}{(M-1)!}\sum_{k=0}^{M}(-1)^k\binom{M}{k}(t-k)_+^{M-1},$$

for $0 \le t \le M$, and zero otherwise. We let $g(t) = s_M(t+\frac{M}{2})$; so, for $M=1$, g is the characteristic function of the interval $[-\frac{1}{2},\frac{1}{2}]$, for $M=2$, g is the "tent" function $g(t) = (1-|t|)_+$.
In [6] the authors observe that, as a consequence of their result, B-splines cannot generate a Gabor frame for values of the time and frequency parameters $a > 0$ and b integer greater than 1. We point out that this has already been proved in [4]. Indeed, in that paper the author established that if $M > 1$ then the family (s_M, a, b) is a Gabor frame for pairs (a,b) such that $a < M$ and $b \le \frac{1}{M}$, and it is not a frame in the following region

$$R = \big\{(a,b) : a > 0, b \in \mathbf{N} \setminus \{1\}\big\} \cup \big\{(a,b) : a > 0, b \ge M\big\}.$$

A similar result holds for the spline of order 1 [4]. In the same paper the remaining region of the time-frequency plane is investigated numerically for the spline of order 2, comparing Daubechies frame bounds estimates with Ron and Shen sharper estimates [3], [10]. Moreover the dual of s_M, $M=2$ is computed for some values of the time and frequency parameters.

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