FRAMES AND OVERSAMPLING FORMULAS FOR BAND LIMITED FUNCTIONS

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Abstract. In this article we obtain families of frames for the space B_ω of functions with band in $[-\omega, \omega]$ by using the theory of shift-invariant spaces. Our results are based on the Gramian analysis of A. Ron and Z. Shen and a variant, due to Bownik, of their characterization of families of functions whose shifts form frames or Riesz bases. We give necessary and sufficient conditions for the translates of a finite number of functions (generators) to be a frame or a Riesz basis for B_ω. We also give explicit formulas for the dual generators and we apply them to Hilbert transform sampling and derivative sampling. Finally, we provide numerical experiments which support the theory.

1. Introduction

In many signal and image processing applications, images and signals are assumed to be band limited. A band limited signal is a function which belongs to the space B_ω of functions in $L^2(\mathbb{R})$ whose Fourier transforms have support in $[-\omega, \omega]$. Functions belonging to this space can be represented by the Whittaker-Kotelnikov-Shannon series, which is the expansion in terms of the orthonormal basis of translates of the sinc function. The coefficients of the expansion are the samples of the function at a uniform grid on $\mathbb{R}$, with "density" ω/π. This sampling density is usually called the *Nyquist* density.

The theory has been extended in many directions by several authors. In one of these extensions the sinc orthonormal basis was replaced by Riesz bases (see for example [Hi] by J. R. Higgins) or frames. Frames generally are overcomplete and their expansion coefficients are not unique. Their redundancy is useful in applications because the reconstruction is more stable with respect to errors in the calculation of the coefficients and it allows the recovery of missing samples [F]. In signal analysis frames can be viewed as an "oversampling" with respect to the Nyquist density.

The second extension consist in using more than one function to generate the space B_ω. In this case the Riesz basis or the frame are formed by the translates of a finite family of functions and the expansion formula is called a multi-channel sampling formula [P] [Hi1].

Finally we mention that there is a huge literature on the problem of reconstruction of signals from non-uniform samples. Since this paper is only concerned with uniform sampling, we refer the reader to the recent article of A. Aldroubi and K. Gröchenig [AG] and the references given there for an extensive review of the problem of non-uniform sampling.

Key words and phrases. Frame, Riesz basis, shift-invariant space, sampling formula, band-limited function.

In this paper we construct multi-channel uniform sampling formulas for band-limited functions using the theory of frames for shift-invariant spaces. A t_o-shift-invariant space is a subspace of $L^2(\mathbb{R})$ that is invariant under all translations $\tau_{kt_o}, k \in \mathbb{Z}$, by integer multiples of a positive number t_o. We recall that B_ω is t_o-shift-invariant for any t_o. A set Φ in a t_o-shift-invariant space S is called a set of generators if S is the closure of the space generated by the family $E_{\Phi,t_o} = \{\tau_{kt_o}\,\varphi,\, \varphi \in \Phi,\, k \in \mathbb{Z}\}$. The space S is said to be finitely generated if it has a finite set of generators. Finitely generated shift-invariant spaces can have different sets of generators; the smallest number of generators is called the length of the space.
The structure of finitely generated shift-invariant spaces was investigated by C. de Boor, R. DeVore and A. Ron with the use of fiberization techniques based on the range function [BDR]. These authors gave conditions under which a finitely generated shift-invariant space has a generating set satisfying properties like stability and orthogonality. Successively, A. Ron and Z. Shen introduced the Gramian analysis and extended the results of [BDR] to countable generated SI spaces [RS]. In their paper they characterized sets of generators whose translates form Bessel sequences, frames and Riesz bases. For finitely generated spaces these conditions are expressed in terms of the eigenvalues of the Gramian matrix. In concrete cases it would be useful to have more explicit conditions expressed in terms of the generators or their Fourier transforms. In this paper, using a result of M. Bownik, we obtain these more explicit conditions for the space B_ω [B]. We also give explicit formulas for the Fourier transforms of the dual generators.
In the last part of the paper we use these results to obtain multichannel over-sampling formulas for band limited signals and we apply them to the derivative sampling. Finally, we support the theory with some numerical experiments.
The paper is organized as follows. In Section 2 we collect some results on frames for shift-invariant spaces and find the range of B_ω as a t_o-shift-invariant space.
In Section 3 we consider a family $\Phi = \{\varphi_1, \varphi_2, \dots \varphi_N\}$ of N generators, where N is the length of B_ω as a t_o-shift-invariant space. We give a necessary and sufficient condition for E_{Φ,t_o} to be a frame or a Riesz basis for B_ω. The condition is expressed in terms of the pre-Gramian. To prove this condition we use a result of Bownik which characterizes the system of translates E_{Φ,t_o} as being a frame or a Riesz family in terms of "fibers" [B].
In Section 4 we find the family Φ^* of dual generators. Here we use the fact that, in the fibered representation of B_ω, the frame operator is unitarily equivalent to the operator of multiplication by the dual Gramian matrix $\widetilde{G}_{\Phi,t_o}(x) = J_{\Phi,t_o}(x) J^*_{\Phi,t_o}(x)$ acting on the fiber over x. The problem of finding the dual generators is reduced to solving the matricial equation $J_{\Phi,t_o}(x) = J_{\Phi,t_o}(x) J^*_{\Phi,t_o}(x)\, J_{\Phi^*,t_o}(x)$ in the unknown $J_{\Phi^*,t_o}(x)$ in $[0,h]$. Where $J_{\Phi,t_o}(x)$ is invertible, the solution is the inverse of $J_{\Phi,t_o}(x)$, elsewhere it is given by the Moore–Penrose inverse of $J_{\Phi,t_o}(x)$. We give the expressions of the Fourier transforms of the dual generators as cross products of translates of the vector $\widehat{\Phi} = (\widehat{\varphi_1}, \widehat{\varphi_2}, \dots \widehat{\varphi_N})$.
In Section 5 we apply the results of the previous sections to obtain two and three-channel sampling formulas for functions of B_ω and we apply them to derivative sampling. We also provide some numerical experiments.

2. Preliminaries

In this section we collect some results on frames for shift-invariant spaces to be used later. We begin by introducing some notation. The Fourier transform of a function f in $L^1(\mathbb{R})$ is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-it\xi} dt.$$

The convolution of two functions f and g is

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int f(x-y) g(y)\, dy,$$

so that $\mathcal{F}(f * g) = \mathcal{F}f\,\mathcal{F}g$. Let h be a positive real number; L^p_h is the space of h-periodic functions on $\mathbb{R}$ such that

$$\|f\|_{L^p_h} = \Big(\frac{1}{h}\int_0^h |f(x)|^p dx\Big)^{1/p} < \infty. \tag{2.1}$$

With the symbol $\ell^2(\mathbb{Z};\mathbb{C}^N)$ we shall denote the space of $\mathbb{C}^N$-valued sequences $c = (c(n))_{\mathbb{Z}}$ such that

$$\|c\|_{\ell_2} = \Big(\sum_{n\in\mathbb{Z}} |c(n)|^2\Big)^{1/2} < \infty.$$

Let H be a subspace of $L^2(\mathbb{R})$. Given a subset $\Phi = \{\varphi_j, j = 1, \dots, N\}$ of H and a positive number t_o denote by E_{Φ,t_o} the set

$$E_{\Phi,t_o} = \{\tau_{nt_o}\varphi_j,\ n \in \mathbb{Z},\ j = 1, \dots, N\}.$$

Here $\tau_a f(x) = f(x+a)$. The closure of the space generated by E_{Φ,t_o} will be denoted by S_{Φ,t_o}. The family E_{Φ,t_o} is a frame for H if the operator $T_{\Phi,t_o} : \ell^2(\mathbb{Z};\mathbb{C}^N) \to H$ defined by

$$T_{\Phi,t_o} c = \sum_{j=1}^N \sum_{n\in Z} c_j(n) \tau_{nt_o}\varphi_j$$

is continuous, surjective and $\operatorname{ran}(T_{\Phi,t_o})$ is closed. The family E_{Φ,t_o} is a frame for H if and only if there exist two constants $0 < A \le B$ such that

$$A\|f\|^2 \le \sum_{j=1}^N \sum_{n\in\mathbb{Z}} |\langle f, \tau_{nt_o}\varphi_j\rangle|^2 \le B\|f\|^2 \qquad \forall f \in H. \tag{2.2}$$

The constants A and B are called frame bounds. If $A = B$ the frame is called *tight* and if $A = B = 1$ a *Parseval* frame. Denote by $T^*_{\Phi,t_o} : H \to \ell^2(\mathbb{Z};\mathbb{C}^N)$ the adjoint of T_{Φ,t_o}, defined by

$$(T^*_{\Phi,t_o} f)_j(n) = \langle f, \tau_{nt_o}\varphi_j\rangle \qquad \forall\, n \in \mathbb{Z},\ j = 1, \dots, N. \tag{2.3}$$

The operator $T_{\Phi,t_o}T^*_{\Phi,t_o} : H \to H$ is called *frame operator.* The set E_{Φ,t_o} is a frame for H if and only if the frame operator is continuously invertible and

$$T_{\Phi,t_o}T^*_{\Phi,t_o} f = \sum_{j=1}^N \sum_{n\in\mathbb{Z}} \langle f, \tau_{nt_o}\varphi_j\rangle \tau_{nt_o}\varphi_j \qquad \forall f \in H.$$

Observe that (2.2) can be written $AI \le\ T_{\Phi,t_o}T^*_{\Phi,t_o}\ \le BI$, where I is the identity operator on H. Denote by Φ^* the family $\Phi^* = \{\varphi^*_j, j = 1, \dots, N\}$, where

$$\varphi^*_j = (T_{\Phi,t_o}T^*_{\Phi,t_o})^{-1}\varphi_j \qquad j = 1, \dots, N. \tag{2.4}$$

If E_{Φ,t_o} is a frame for H then E_{Φ^*,t_o} is also a frame (the *dual frame*), and $T_{\Phi,t_o}T^*_{\Phi^*,t_o} = T_{\Phi^*,t_o}T^*_{\Phi,t_o} = I$. Explicitly

$$f = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}\langle f, \tau_{nt_o}\varphi_j^*\rangle\tau_{nt_o}\varphi_j = \sum_{j=1}^{N}\sum_{n\in\mathbb{Z}}\langle f, \tau_{nt_o}\varphi_j\rangle\tau_{nt_o}\varphi_j^* \qquad \forall f \in H. \tag{2.5}$$

The elements of Φ are called *generators* and the elements of Φ^* *dual generators.* If the family E_{Φ,t_o} is a frame for H and the operator T_{Φ,t_o} is injective, then E_{Φ,t_o} is called a *Riesz basis.*

In what follows t_o is a positive parameter. To simplify notation, throughout the paper we shall set

$$h = \frac{2\pi}{t_o}.$$

A subspace S of $L^2(\mathbb{R})$ is t_o-shift-invariant if it is invariant under all translations by a multiple of t_o. The following bracket product plays an important role in Ron and Shen's analysis of shift-invariant spaces. For f and $g \in L^2(\mathbb{R})$, define

$$[f, g] = h\sum_{j\in\mathbb{Z}} f(\cdot + jh)\overline{g}(\cdot + jh). \tag{2.6}$$

Note that $[f, g]$ is in L^1_h and $\|[f, f]\|_{L^1_h} = \|f\|_2^2$. The Fourier coefficients of $[\hat{f}, \hat{g}]$ are given by

$$[\hat{f}, \hat{g}]\widehat{\ }(\ell) = \langle f, \tau_{\ell t_o} g\rangle \qquad \forall\, \ell \in \mathbb{Z}. \tag{2.7}$$

Indeed

$$[\hat{f}, \hat{g}]\widehat{\ }(\ell) = \int_0^h \sum_j \tau_{jh}(\hat{f}\,\overline{\hat{g}})(x)e^{-2\pi i\ell\frac{x}{h}}\,dx = \int \hat{f}(x)\,\overline{\hat{g}}(x)\; e^{-i\ell t_o x}dx$$

If S is a t_o-shift-invariant space and there exists a finite family Φ such that $S = S_{\Phi,t_o}$, then we say that S is finitely generated. Riesz bases for finitely generated shift-invariant spaces have been studied by various authors. In [BDR] the authors gave a characterization of such bases. In [RS1] Ron and Shen gave a characterization of frames and tight frames also for countable sets Φ. The principal notions of their theory are the *pre-Gramian*, the *Gramian* and the *dual Gramian* matrices.

The *pre-Gramian* J_{Φ,t_o} is the h-periodic function mapping $\mathbb{R}$ to the space of $\infty\times N$-matrices defined on $[0, h]$ by

$$\big(J_{\Phi,t_o}\big)_{j\ell}(x) = \sqrt{h}\;\widehat{\varphi_\ell}(x + jh) \qquad \forall j \in \mathbb{Z},\, \ell = 1, \ldots, N. \tag{2.8}$$

The pre-Gramian J_{Φ,t_o} should not be confused with the matrix-valued function whose entries are $\sqrt{h}\,\widehat{\varphi_\ell}(x+jh)$, for all $x \in \mathbb{R}$, which is not periodic. The spectrum of the space S_{Φ,t_o} is defined as

$$\sigma(S_{\Phi,t_o}) = \{x \in \mathbb{R} : J_{\Phi,t_o}(x) \neq 0\} \tag{2.9}$$

or, equivalently, as the support of $\sum_{j=1}^N[\widehat{\varphi_j}, \widehat{\varphi_j}]$. Of course, since the functions $\widehat{\varphi_j}$ are defined only up to a null-set, the support is intended in the sense of distributions, i.e. as the complement of the largest open set on which the function $\sum_{j=1}^N[\widehat{\varphi_j}, \widehat{\varphi_j}]$ vanishes as distribution. It was proved in [BDR] that the spectrum of a finitely generated space depends only on the space itself and not on the particular selection of its generators.

Denote by J^*_{Φ,t_o} the adjoint of J_{Φ,t_o}. The *Gramian* matrix $G_{\Phi,t_o} = J^*_{\Phi,t_o} J_{\Phi,t_o}$ is the $N \times N$ matrix whose elements are the h-periodic functions

$$\big(G_{\Phi,t_o}\big)_{j\ell} = [\widehat{\varphi_\ell}, \widehat{\varphi_j}]. \tag{2.10}$$

The *dual Gramian* $\widetilde{G}_{\Phi,t_o} = J_{\Phi,t_o} J^*_{\Phi,t_o}$ is the infinite matrix whose elements are

$$\big(\widetilde{G}_{\Phi,t_o}\big)_{j\ell} = h \sum_{n=1}^{N} \tau_{jh}\widehat{\varphi_n}\ \tau_{\ell h}\overline{\widehat{\varphi_n}}, \qquad j,\ell \in \mathbb{Z}. \tag{2.11}$$

The importance of these two matrices lies in the fact that the Gramian matrix represents the operator $T^*_{\Phi,t_o} T_{\Phi,t_o}$ and the dual Gramian represents the operator $T_{\Phi,t_o} T^*_{\Phi,t_o}$ and many properties of these operators can be studied by looking at them. Indeed, by the theory of Ron and Shen [RS], after conjugating with an isometry, the operator $T_{\Phi,t_o} T^*_{\Phi,t_o}$ can be decomposed into a measurable field of operators, acting on $\ell^2(\mathbb{Z})$, which are represented by the dual Gramian matrix $\widetilde{G}_{\Phi,t_o}$ in the canonical basis (see formulas (3.4) and (3.5) below). Similarly, after conjugation with a Fourier transform, the operator $T^*_{\Phi,t_o} T_{\Phi,t_o}$ is represented by the Gramian G_{Φ,t_o}.

Denote by $L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$ the Hilbert space of $\ell^2(\mathbb{Z})$-valued h-periodic functions on $\mathbb{R}$ such that

$$\|f\|_{L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))} = \Big(\frac{1}{h}\int_0^h \|f(x)\|_{\ell^2}\ dx\Big)^{\frac{1}{2}} < \infty. \tag{2.12}$$

For every f in $L^2(\mathbb{R})$ we denote by $\mathcal{L}_h f$ the $\ell^2(\mathbb{Z})$-valued function defined on $[0,h]$ by

$$\mathcal{L}_h f(x) = \sqrt{h}\sum_{l\in\mathbb{Z}} f(x+\ell h)\delta_\ell \qquad x \in [0,h] \tag{2.13}$$

and extended to $\mathbb{R}$ as a periodic function of period h. Here $\{\delta_\ell : \ell \in \mathbb{Z}\}$ is the canonical basis of $\ell^2(\mathbb{Z})$. The map $f \mapsto \mathcal{L}_h f$ is an isometry of $L^2(\mathbb{R})$ onto $L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$. Observe that the vectors $\mathcal{L}_h\widehat{\varphi_j},\ j = 1,\dots,N$ are the columns of the pre-Gramian J_{Φ,t_o}, i.e.

$$J_{\Phi,t_o} = \big(\mathcal{L}_h\widehat{\varphi_1},\dots,\mathcal{L}_h\widehat{\varphi_N}\big). \tag{2.14}$$

The map $\mathcal{L}_h$ links t_o-shift-invariant subspaces of $L^2(\mathbb{R})$ with h-doubly-invariant subspaces of $L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$. We recall that a subspace of $L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$ is h-doubly-invariant if it is invariant under pointwise multiplication by $e^{2\pi i k\frac{x}{h}}$, $k \in \mathbb{Z}$. Obviously a subspace S of $L^2(\mathbb{R})$ is t_o-shift-invariant if and only if the space

$$\mathcal{L}_h(\hat{S}) = \{\mathcal{L}_h\hat{f}, f \in S\} \tag{2.15}$$

is h-doubly-invariant. T.P. Srinivanasan gave a characterization of doubly-invariant spaces in terms of *range functions* (see [H],[S]). As remarked by de Boor, DeVore and Ron [BDR] a similar characterization of shift-invariant spaces follows from it (see Proposition 2.1 below). In our context a range function is a h-periodic map $\mathcal{R}$ from $\mathbb{R}$ to the closed subspaces of $\ell^2(\mathbb{Z})$. The function $\mathcal{R}$ is measurable if the map $\mathcal{P}$ which maps a point $x \in \mathbb{R}$ to the orthogonal projection $\mathcal{P}(x)$ onto $\mathcal{R}(x)$ is weakly measurable in the operator sense, i.e. the function

$$x \mapsto (P(x)\varphi,\psi)_{\ell^2(\mathbb{Z})}$$

is measurable for all φ and $\psi \in \ell^2(\mathbb{Z})$. Range functions which are equal almost everywhere are identified.

Proposition 2.1. *A closed subspace S of $L^2(\mathbb{R})$ is t_o-shift-invariant if and only if*

$$S = \{f \in L^2(\mathbb{R}) : \mathcal{L}_h \hat{f}(x) \in \mathcal{R}(x) \text{ for } a.e.\, x \in \mathbb{R}\} \tag{2.16}$$

for some measurable h-periodic range function $\mathcal{R}$.

Obviously t_o-shift-invariant subspaces with the same range function coincide. This observation shall be used in the proof of Theorems 3.6 and 3.7.
In Theorem 2.2 below we compute the range function of the space of band-limited functions

$$B_\omega = \{f \in L^2(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subset [-\omega, \omega]\}.$$

In general it is not a simple matter to compute the range function of a space. Fortunately, if the space S is finitely generated, the range function can be written in terms of the generators; indeed in [BDR] it was shown that if $S = S_{\Phi, t_o}$ and $\Phi = \{\varphi_j, \quad j = 1, \dots N\}$ then

$$\mathcal{R}_h(S_{\Phi,t_o})(x) = \mathrm{span}\{\mathcal{L}_h \widehat{\varphi_j}(x) : j = 1, \dots, N\} \tag{2.17}$$

is the space generated by the columns of the pre-Gramian matrix J_{Φ,t_o}. This result holds also for countable sets of generators. Of course a shift-invariant space S can have more than one family of generators; the smallest number of generators is called the length of the space

$$\mathrm{len}_{t_o}(S) = \min\{\sharp\Phi, S = S_{\Phi,t_o}\}. \tag{2.18}$$

In [BDR] it has been proved that if S is finitely generated then

$$\mathrm{len}_{t_o}(S) = \operatorname{ess\,sup}\big\{\dim \mathcal{R}_h(S_{\Phi,t_o})(x) : \quad x \in [0, h]\big\}. \tag{2.19}$$

To state the next theorem we need some notation. We denote by $\lfloor a \rfloor$ the greatest integer less than a. Recall that $h = 2\pi/t_o$ and set $\ell = \lfloor \omega/h \rfloor + 1$.
If $\omega/\ell \le h < \omega/(\ell - 1/2)$ then $0 \le -\omega + \ell h < \omega - (\ell-1)h < h$. We denote by I_-, I, I_+ the intervals defined by

$$I_- = (0, -\omega + \ell h), \qquad I = (-\omega + \ell h, \omega - (\ell - 1)h), \qquad I_+ = (\omega - (\ell-1)h, h) \tag{2.20}$$

Similarly if $\omega/(\ell - 1/2) \le h < \omega/(\ell - 1)$ then $0 < \omega - (\ell-1)h \le -\omega + \ell h < h$; in this case we denote by K_-, K, K_+ the intervals defined by
(2.21)

$$K_- = (0, \omega - (\ell-1)h), \qquad K = (\omega - (\ell - 1)h, -\omega + \ell h), \qquad K_+ = (-\omega + \ell h, h).$$

Recall that $\{\delta_\ell : \ell \in \mathbb{Z}\}$ is the canonical basis of $\ell^2(\mathbb{Z})$.

Theorem 2.2. *Let t_o be a positive parameter, $h = 2\pi/t_o$ and set $\ell = \lfloor \omega/h \rfloor + 1$. Then*

(i) *if $\omega/\ell \le h < \omega/(\ell - 1/2)$ the range function of the space B_ω is*

$$\mathcal{R}_h(B_\omega)(x) = \begin{cases} \mathrm{span}\{\delta_j : \ -(\ell - 1) \le j \le \ell - 1\} & \text{if} \quad x \in I_- \\ \mathrm{span}\{\delta_j : \ -\ell \le j \le \ell - 1\} & \text{if} \quad x \in I \\ \mathrm{span}\{\delta_j : \ -\ell \le j \le \ell - 2\} & \text{if} \quad x \in I_+. \end{cases} \tag{2.22}$$

Note that if $h = \omega/\ell$ then the intervals I_- and I_+ are empty.

(ii) *If $\omega/(\ell-1/2) \le h < \omega/(\ell-1)$ the range function of the space B_ω is*

$$\mathcal{R}_h(B_\omega)(x) = \begin{cases} \operatorname{span}\{\delta_j : \ -(\ell-1) \le j \le \ell-1\} & \text{if} \quad x \in K_- \\ \operatorname{span}\{\delta_j : \ -(\ell-1) \le j \le \ell-2\} & \text{if} \quad x \in K \\ \operatorname{span}\{\delta_j : \ -\ell \le j \le \ell-2\} & \text{if} \quad x \in K_+. \end{cases} \tag{2.23}$$

Note that if $h = \omega/(\ell-1/2)$ then the interval K is empty.

Proof. For the sake of simplicity we prove the theorem only for $\ell = 2$, i.e. $\omega/2 \le h < \omega$. The proof in the other cases is similar. Denote by $\mathcal{M}$ the space of all functions g in $L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$ such that $g(x) \in \mathcal{R}(B_\omega)(x)$ for a.e. $x \in [0,h]$. Let $\mathbf{Q}$ denote the orthogonal projection of $L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$ onto $\mathcal{M}$. By [H, Lemma p. 58] for $g \in L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$ we have

$$\mathbf{Q}g(x) = Q(x)g(x) \qquad \textit{a.e. } x \tag{2.24}$$

where $Q(x)$ is the orthogonal projection of $\ell^2(\mathbb{Z})$ onto $\mathcal{R}(B_\omega)(x)$. Thus, to determine $\mathcal{R}(B_\omega)(x)$ we only need to describe the projection $Q(x)$ for a.e. x. Denote by Λ_ω the space of functions in $L^2(\mathbb{R})$ with support in $[-\omega,\omega]$ and by $\mathbf{P} : L^2(\mathbb{R}) \mapsto \Lambda_\omega$ the orthogonal projection onto Λ_ω, that is $\mathbf{P}f = \chi_{[-\omega,\omega]}f$. By Proposition 2.1 $\mathcal{L}_h(\Lambda_\omega) = \mathcal{L}_h(\widehat{B_\omega}) = \mathcal{M}$. Thus we have

$$\mathbf{Q} = \mathcal{L}_h \mathbf{P} \mathcal{L}_h^{-1} \tag{2.25}$$

because $\mathcal{L}_h$ is an isometry. Let $\psi = \sum_{n\in\mathbb{Z}} \psi_n \delta_n \in L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$, then $\mathcal{L}_h^{-1}\psi(x) = \psi_n(x)$ for $x \in [nh,(n+1)h]$, $n \in \mathbb{Z}$; i.e.

$$\mathcal{L}_h^{-1}\psi = \sum_n \chi_{[nh,(n+1)h]}\psi_n.$$

Hence

$$\mathbf{P}\mathcal{L}_h^{-1}\psi = \sum_n \chi_{[-\omega,\omega]}\chi_{[nh,(n+1)h]}\psi_n. \tag{2.26}$$

Since $\frac{\omega}{2} \le h < \omega$

$$\chi_{[-\omega,\omega]}\chi_{[nh,(n+1)h]} = \begin{cases} \chi_{[0,h]} & \text{if} \quad n = 0 \\ \chi_{[-h,0]} & \text{if} \quad n = -1 \\ \chi_{[h,\omega]} & \text{if} \quad n = 1 \\ \chi_{[-\omega,-h]} & \text{if} \quad n = -2 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\mathbf{P}\mathcal{L}_h^{-1}\psi = \chi_{[-\omega,-h]}\,\psi_{-2} + \chi_{[-h,0]}\,\psi_{-1} + \chi_{[0,h]}\psi_0 + \chi_{[h,\omega]}\,\psi_1. \tag{2.27}$$

Now we find $\mathcal{L}_h\mathbf{P}\mathcal{L}_h^{-1}\psi(x)$.
First suppose that $h \ge 2\omega/3$; then $0 < \omega - h \le 2h - \omega < h$. Hence from (2.27) it follows that for each $\psi \in \ell^2(\mathbb{Z})$

$$\mathcal{L}_h\mathbf{P}\mathcal{L}_h^{-1}\psi(x) = \begin{cases} \psi_{-1}(x)\,\delta_{-1} + \psi_0(x)\,\delta_0 + \psi_1(x)\,\delta_1 & \text{if} \quad x \in \ (0,\omega-h) \\ \psi_{-1}(x)\,\delta_{-1} + \psi_0(x)\,\delta_0 & \text{if} \quad x \in \ (\omega-h, 2h-\omega) \\ \psi_{-2}(x)\,\delta_{-2} + \psi_{-1}(x)\,\delta_{-1} + \psi_0(x)\,\delta_0 & \text{if} \quad x \in \ (2h-\omega, h). \end{cases}$$

Suppose now $h < 2\omega/3$; in this case $2h-\omega < \omega - h < h$. Hence from (2.27) we have $\psi \in \ell^2(\mathbb{Z})$

$$\mathcal{L}_h \mathbf{P} \mathcal{L}_h^{-1}\psi(x) = \begin{cases} \psi_{-1}(x)\,\delta_{-1} + \psi_0(x)\,\delta_0 + \psi_1(x)\,\delta_1 & \text{if} \quad x \in (0, 2h-\omega) \\ \displaystyle\sum_{j=-2}^{1} \psi_j(x)\,\delta_j & \text{if} \quad x \in (2h-\omega, \omega - h) \\ \psi_{-2}(x)\,\delta_{-2} + \psi_{-1}(x)\,\delta_{-1} + \psi_0(x)\,\delta_0 & \text{if} \quad x \in (\omega - h, h). \end{cases}$$

Note that if $h = \omega/2$ the intervals $(0, 2h-\omega)$ and $(\omega - h, h)$ are empty. This determines completely the projection $Q(x)$ by (2.24) and (2.25). □

The following corollary is a straightforward consequence of Theorem 2.2.

Corollary 2.3. *Let $\ell = \lfloor \omega/h \rfloor + 1$. Then the length of B_ω, as t_o-shift-invariant space, is*

$$\operatorname{len}_{t_o}(B_\omega) = \begin{cases} 2\ell & \text{if } \frac{\omega}{\ell} \le h < \frac{\omega}{\ell - 1/2}, \\ 2\ell - 1 & \text{if } \frac{\omega}{\ell-1/2} \le h < \frac{\omega}{\ell-1}. \end{cases} \tag{2.28}$$

Proof. The thesis follows immediately from (2.19) and Theorem 2.2. □

3. Frames for the space B_ω

Let Φ be a finite family of generators for B_ω. In this section we find conditions under which E_{Φ,t_o} is a frame for B_ω. First we prove a representation formula for the frame operator analogous to the formula proved by Heil and Walnut for Gabor frames [HW, Theorem 4.2.1]. From this formula we deduce a simple necessary condition (see Proposition 3.2). The formula will also be useful in Section 4 to find the dual generators.

Let $\Phi = \{\varphi_j, 1 \le j \le N\}$ be a family of functions in B_ω, t_o a positive parameter and $h = 2\pi/t_o$. Denote by Ω^k_{Φ,t_o} the function

$$\Omega^k_{\Phi,t_o} = h \sum_{j=1}^{N} \widehat{\varphi_j} \tau_{kh} \overline{\widehat{\varphi_j}} \qquad k \in \mathbb{Z}. \tag{3.1}$$

Theorem 3.1. *If $\sum_k \|\Omega^k_{\Phi,t_o}\|_\infty < \infty$ then the operator T_{Φ,t_o} from $\ell^2(\mathbb{Z};\mathbb{C}^N)$ to B_ω is bounded and*

$$\mathcal{F} T_{\Phi,t_o} T^*_{\Phi,t_o} \mathcal{F}^{-1} = \sum_k \Omega^k_{\Phi,t_o} \tau_{kh} \tag{3.2}$$

on $L^2(\mathbb{R})$, where the series converges in the operator norm.

Proof. First we show that if $f \in L^2(\mathbb{R})$ then $[\widehat{f}, \widehat{\varphi_j}]$ is in L^2_h for $1 \le j \le N$. Indeed

$$\begin{aligned} \int_0^h \big|[\widehat{f}, \widehat{\varphi_j}]\big|^2 dx &= h \int_0^h \Big(\sum_\ell \overline{\widehat{f}}(x + \ell h)\, \widehat{\varphi_j}(x+\ell h) \Big) [\widehat{f}, \widehat{\varphi_j}](x) dx \\ &= h \sum_\ell \int_{\ell h}^{(\ell+1)h} \overline{\widehat{f}}(z)\, \widehat{\varphi_j}(z)\, [\widehat{f}, \widehat{\varphi_j}](z)\, dz \\ &= h^2 \int \sum_k \widehat{\varphi_j}(z)\, \overline{\widehat{\varphi}}(z + kh)\, \widehat{f}(z+kh) \overline{\widehat{f}}(z)\, dz. \end{aligned}$$

Note that we may exchange the sum and the integral because $\overline{\widehat{f}}\,\widehat{\varphi_j}$ has compact support and the sum is finite. By summing over j and exchanging the sums we obtain

$$\sum_{j=1}^{N}\frac{1}{h}\int_0^h \big|[\hat{f},\widehat{\varphi_j}]\big|^2 dx = h\int\sum_k\sum_{j=1}^{N}\widehat{\varphi_j}(z)\overline{\widehat{\varphi}}(z+kh)\widehat{f}(z+kh)\overline{\widehat{f}}(z)\,dz.$$

Therefore, by (3.1)

$$\sum_{j=1}^{N}\frac{1}{h}\int_0^h \big|[\widehat{f},\widehat{\varphi_j}]\big|^2 dx = \langle\sum_k \Omega^k_{\Phi,t_o}\tau_{kh}\widehat{f},\widehat{f}\rangle. \tag{3.3}$$

By Schwarz's inequality the right hand side is less then $\|\hat{f}\|_2^2\sum_k\|\Omega^k_{\Phi,t_o}\|_\infty$. Hence $[\widehat{f},\hat{\varphi}_j]\in L^2_h$, for $1\le j\le N$. By (2.7) the Fourier coefficients of $[\widehat{f},\widehat{\varphi_j}]$ are $\langle f,\tau_{kh}\varphi_j\rangle$, $k\in\mathbb{Z}$. Hence, by (3.3), Plancherel's formula and (2.3)

$$\begin{aligned}\langle\sum_k \Omega^k_{\Phi,t_o}\tau_{kh}\widehat{f},\widehat{f}\rangle &= \sum_{j=1}^{N}\sum_k|\langle f,\tau_{kh}\varphi_j\rangle|^2\\ &= \|T^*_{\Phi,t_o}f\|^2\\ &= \langle T_{\Phi,t_o}T^*_{\Phi,t_o}f,f\rangle.\end{aligned}$$

This proves (3.2). Hence T^*_{Φ,t_o} is bounded from B_ω to $\ell^2(\mathbb{Z};\mathbb{C}^N)$ and T_{Φ,t_o} is bounded from $\ell^2(\mathbb{Z};\mathbb{C}^N)$ to B_ω. $\square$

Now by using formula (3.2) we show that the operator $T_{\Phi,t_o}T^*_{\Phi,t_o}$ is unitarily equivalent to the operator of multiplication by the matrix $\widetilde{G}_{\Phi,t_o}$ acting on the space $L^2_h(\mathbb{R};\ell^2(\mathbb{Z}))$. Our proof, given for the sake of completeness, is a simple alternative derivation of a result of Ron and Shen for general shift-invariant spaces [RS].
Let f be a function in $L^2(\mathbb{R})$. By (3.2) for each $j\in\mathbb{Z}$

$$\begin{aligned}\mathcal{F}T_{\Phi,t_o}T^*_{\Phi,t_o}\mathcal{F}^{-1}f(x+jh) &= \sum_k\Omega^k_{\Phi,t_o}(x+jh)f(x+jh+kh)\\ &= \sum_\ell\Omega^{\ell-j}_{\Phi,t_o}(x+jh)f(x+\ell h).\end{aligned}$$

Now we observe that for a.e. x in $\mathbb{R}$

$$\Omega^{\ell-j}_{\Phi,t_o}(x+jh)=\overline{\Omega^{j-\ell}_{\Phi,t_o}}(x+\ell h)\qquad\forall\, j,\ell\in\mathbb{Z}.$$

Therefore by (2.11) and (3.1) we get

$$\mathcal{F}T_{\Phi,t_o}T^*_{\Phi,t_o}\mathcal{F}^{-1}\,f(x+jh)=\sum_\ell\big(\widetilde{G}_{\Phi,t_o}\big)_{j\ell}(x)\;f(x+\ell h)\qquad\forall\, j,\ell\in\mathbb{Z}.$$

By (2.13) this formula implies that

$$\mathcal{L}_h\mathcal{F}T_{\Phi,t_o}T^*_{\Phi,t_o}\mathcal{F}^{-1}f=\widetilde{G}_{\Phi,t_o}\mathcal{L}_hf\qquad\text{a.e. in }[0,h]. \tag{3.4}$$

This shows that the operator $T_{\Phi,t_o}T^*_{\Phi,t_o}$ is unitarily equivalent to the operator $\widetilde{\mathcal{G}}_{\Phi,t_o}$ defined by

$$\widetilde{\mathcal{G}}_{\Phi,t_o}g(x)=\widetilde{G}_{\Phi,t_o}(x)g(x) \tag{3.5}$$

for almost every $x \in [0, h]$. An operator of this form is said to be decomposable into the measurable field $x \mapsto \widetilde{\mathcal{G}}_{\Phi,t_o}(x)$ of operators on $l^2(\mathbb{Z})$. We shall use this representation in Section 3 to find the dual generators of frames.

In the following proposition we give a necessary condition for E_{Φ,t_o} to be a frame. The proof mimics closely an argument of Heil and Walnut for Gabor frames [HW].

Proposition 3.2. *Let $\Phi = \{\varphi_j, 1 \le j \le N\}$ be a family of functions of B_ω. If E_{Φ,t_o} is a frame for B_ω then there exist $0 < \delta \le \gamma < \infty$ such that*

$$\delta \le \sum_{j=1}^{N} |\widehat{\varphi_j}|^2 \le \gamma \qquad \text{a.e. in } [-\omega, \omega]. \tag{3.6}$$

Proof. Observe that $h \sum_{j=1}^{N} |\widehat{\varphi_j}|^2 = \Omega^0_{\Phi,t_o}$. We only prove that $0 < \operatorname{ess\,inf} \Omega^0_{\Phi,t_o}$ because the proof of the other inequality is analogous. Suppose that $\operatorname{ess\,inf} \Omega^0_{\Phi,t_o} = 0$; then for each ϵ there exists a set $E_\epsilon \subset (\omega, \omega)$, of positive measure, such that $\Omega^0_{\Phi,t_o}(x) < \epsilon$ for a.e. $x \in E_\epsilon$. We may suppose that there exists an an interval I of measure h such that $E_\epsilon \subset I$. Let $f \in B_\omega$ be defined by $\hat{f} = \chi_{E_\epsilon}$; by the Parseval and the Plancherel formula

$$\sum_k |\langle f, \tau_{kt_o}\varphi_j\rangle|^2 = \sum_k \Big| \int_I \widehat{f}\, \overline{\widehat{\varphi_j}} e^{2\pi i k \frac{x}{h}}\, dx \Big|^2 = h \int_I \chi_{E_\epsilon} \big|\widehat{\varphi_j}\big|^2\, dx \qquad j = 1, \ldots, N.$$

Therefore

$$\sum_{j=1}^{N} \sum_k |\langle f, \tau_{kt_o}\varphi_j\rangle|^2 = \int_I \chi_{E_\epsilon}(x) \Omega^0_{\Phi,t_o}(x)\, dx < \epsilon h \|\chi_{E_\epsilon}\|^2 = \epsilon h \|f\|^2.$$

This contradicts the fact that E_{Φ,t_o} is a frame. $\square$

Next, we give a necessary and sufficient conditions for E_{Φ,t_o} to be a frame or a Riesz basis for B_ω, when Φ is a subset of B_ω of cardinality $\operatorname{len}_{t_o}(B_\omega)$ (see Theorems 3.6 and 3.7 below). Our characterization will be given in terms of the pre-Gramian. Strictly speaking, the pre-Gramian $J_{\Phi,to}$ is an infinite matrix. However we shall see that all but a finite number of the rows of $J_{\Phi,to}$ vanish. Hence we may identify it with a finite matrix. We shall need the following

Lemma 3.3. *Let h be a positive number, $\ell = \lfloor \omega/h \rfloor + 1$ and $g \in B_\omega$. Let I_-, I, I_+ and K_-, K, K_+ be the intervals defined in the previous section (see (2.20) and (2.21)).*

(i) *Suppose that $\omega/\ell \le h < \omega/(\ell - 1/2)$. Then $\tau_{jh}\, g(x) = 0$*

$$\begin{aligned} &if\ x \in I_- \quad and\ j \notin \{-(\ell-1) \le j \le \ell - 1\} \\ &if\ x \in I \quad\ \ and\ j \notin \{-\ell \le j \le \ell - 1\} \\ &if\ x \in I_+ \quad and\ j \notin \{-\ell \le j \le \ell - 2\}. \end{aligned} \tag{3.7}$$

Note that if $h = \omega/\ell$ then I_- and I_+ are empty.

(ii) *Suppose that $\omega/(\ell - 1/2) \le h < \omega/(\ell - 1)$. Then $\tau_{jh}\, g(x) = 0$*

$$\begin{aligned} &if\ x \in K_- \quad and\ j \notin \{-(\ell-1) \le j \le \ell - 1\} \\ &if\ x \in K \quad\ \ and\ j \notin \{-(\ell-1) \le j \le \ell - 2\} \\ &if\ x \in K_+ \quad and\ j \notin \{-\ell \le j \le \ell - 2\}. \end{aligned} \tag{3.8}$$

Note that if $h = \omega/(\ell - 1/2)$ then K is empty.

We omit the proof which is straightforward.

We consider separately the two cases $h < \omega/(\ell - 1/2)$ and $\omega/(\ell - 1/2) \le h$. Assume first that $\omega/\ell \le h < \omega/(\ell - 1/2)$. Then $\mathrm{len}_{t_o}(B_\omega) = 2\ell$ by Corollary 2.3. Let $\Phi = \{\varphi_j : 1 \le j \le 2\ell\}$ be a subset of B_ω of cardinality 2ℓ. By Lemma 3.3 all the rows of the matrix J_{Φ,t_o} vanish except possibly $\big(\tau_{jh}\widehat{\varphi}_1, \tau_{jh}\widehat{\varphi}_2, \ldots, \tau_{jh}\widehat{\varphi}_{2\ell}\big)$, $-\ell \le j \le \ell - 1$. Thus we identify the infinite matrices J_{Φ,t_o}, J^*_{Φ,t_o} and G_{Φ,t_o} with their $2\ell \times 2\ell$ submatrices corresponding to these rows. The entries of the Gramian matrix are

$$\big(G_{\Phi,t_o}\big)_{jk} = [\widehat{\varphi_k}, \widehat{\varphi_j}] \qquad 1 \le j \le 2\ell \qquad 1 \le k \le 2\ell.$$

By Lemma 3.3 the i-th column of J_{Φ,t_o}, $1 \le i \le 2\ell$ is

$$\sqrt{h}\begin{bmatrix} 0 \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ \tau_{(\ell-1)h}\widehat{\varphi}_i \end{bmatrix} \text{ in } I_- \qquad \sqrt{h}\begin{bmatrix} \tau_{-\ell h}\widehat{\varphi}_i \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ \tau_{(\ell-1)h}\widehat{\varphi}_i \end{bmatrix} \text{ in } I \qquad \sqrt{h}\begin{bmatrix} \tau_{-\ell h}\widehat{\varphi}_i \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ 0 \end{bmatrix} \text{ in } I_+. \tag{3.9}$$

We note that

$$\operatorname{rank} G_{\Phi,t_o} = \operatorname{rank} J_{\Phi,t_o}. \tag{3.10}$$

We shall use the following result of Bownik [B, Thm 2.3] which characterizes the system of translates E_{Φ,t_o} as being a frame or a Riesz family (for the space it generates) in terms of the "fibers" $\{\mathcal{L}_h\hat{\varphi}(x) : \varphi \in \Phi\}$.

Theorem 3.4. *Suppose $\Phi \subset L^2(\mathbb{R}^n)$ is countable and let H be the subspace of $L^2(\mathbb{R})$ generated by E_{Φ,t_o}. Then*

(i) *E_{Φ,t_o} is a frame for H with constants A, B if and only if $\{\mathcal{L}_h\hat{\varphi}(x) : \varphi \in \Phi\}$ is a frame for $\mathcal{R}_h(H)(x)$ with constants A, B for a.e. $x \in [0, h]$.*

(ii) *E_{Φ,t_o} is a Riesz basis for H with constants A, B if and only if $\{\mathcal{L}_h\hat{\varphi} : \varphi \in \Phi\}$ is a Riesz basis for $\mathcal{R}_h(H)(x)$ with constants A, B for a.e. $x \in [0, h]$.*

To apply Bownik's theorem in our context we need a simple lemma of linear algebra. Let J be a $n \times m$ matrix with complex entries, $n \le m$; we shall denote by $\|J\|$ the norm of J as linear operator from $\mathbb{C}^m$ to $\mathbb{C}^n$ and by $[J]_n$ the sum of the squares of the absolute values of the minors of order n of J.

Lemma 3.5. *Let $v_1, \ldots, v_m$ be m vectors in $\mathbb{C}^n$, $m \ge n$, and denote by J the matrix $(v_1, \ldots, v_m)$ whose j-th column is the vector v_j.*

(i) *If $[J]_n > 0$ then $\{v_1, \ldots, v_m\}$ is a frame of $\mathbb{C}^n$ with frame constants $A \ge [J]_n \|J\|^{2(1-n)}$, $B = \|J\|^2$. Conversely, if $\{v_1, \ldots, v_m\}$ is a frame for $\mathbb{C}^n$ with constants A and B, then $[J]_n \ge A^n$ and $\|J\| \le B^{1/2}$.*

(ii) *If $m = n$ and $\det J > 0$ then $\{v_1, \ldots, v_m\}$ is a Riesz basis of $\mathbb{C}^n$ with constants $A = \det(J)^2 \|J\|^{2(1-n)}$ and $B = \|J\|^2$. Conversely, if $\{v_1, \ldots, v_m\}$ is a Riesz basis for $\mathbb{C}^n$ with constants A and B then $\det(J) \ge A^{n/2}$ and $\|J\| \le B^{1/2}$.*

Proof. Let $T : \mathbb{C}^m \to \mathbb{C}^n$ be the synthesis operator associated to $\{v_1, \dots, v_m\}$, i.e. $Tz = \sum_{i=1}^m z_j v_j$ for all $z \in \mathbb{C}^m$. We observe that TT^* and JJ^* have the same eigenvalues $\lambda_1 \le \cdots \le \lambda_n$, because the matrix J represents the operator T with respect to the canonical bases of $\mathbb{C}^m$ and $\mathbb{C}^n$. Since

$$\lambda_1 I \le TT^* \le \lambda_n I$$

$\{v_1, \dots, v_m\}$ is a frame for $\mathbb{C}^n$ if and only if $\lambda_1 > 0$. In such a case λ_1 is the biggest lower frame bound and λ_n is the smallest upper frame bound. Moreover $\lambda_n = \|JJ^*\| = \|J\|^2$.

Now suppose that $[J]_n > 0$. By the Cauchy-Binet theorem $[J]_n = \det(JJ^*) = \prod_{j=1}^n \lambda_j$. Thus

$$\lambda_1 = \frac{\det(JJ^*)}{\prod_{k=2}^n \lambda_k} \ge \frac{[J]_n}{\lambda_n^{n-1}} = \frac{[J]_n}{\|J\|^{2(n-1)}} > 0$$

and $\{v_1, \dots, v_m\}$ is a frame for $\mathbb{C}^n$ with constants $\lambda_1 \ge [J]_n \|J\|^{2(1-n)}$ and $\lambda_n = \|J\|^2$.

Conversely, suppose that $\{v_1, \dots, v_m\}$ is a frame for $\mathbb{C}^n$ with constants A and B. Then $A \le \lambda_1 \le \lambda_n \le B$. Hence $\|J\| \le B^{1/2}$ and

$$[J]_n = \det(JJ^*) \ge \lambda_1^n \ge A^n > 0.$$

This concludes the proof of part (i) of the Lemma.

To prove the second part it suffices to observe that $\{v_1, \dots, v_n\}$ is a Riesz basis of $\mathbb{C}^n$ if and only if T is an isomorphism and that, in such a case, Riesz constants are also frame bounds. Moreover $[J]_n = |\det J|^2$ when $m = n$. $\square$

Theorem 3.6. *Suppose that $\omega/\ell \le h < \omega/(\ell - \frac{1}{2})$. Let $\Phi = \{\varphi_j, 1 \le j \le 2\ell\}$ be a subset of B_ω. Then E_{Φ,t_o} is a frame for B_ω if and only if there exist positive constants δ, γ, σ and η such that*

$$\delta \le \sum_{j=1}^{2\ell} |\widehat{\varphi}_j|^2 \le \gamma \qquad a.e.\ in \quad (-\omega, \omega), \tag{3.11}$$

$$[\, J_{\Phi,t_o}\,]_{2\ell-1} \ge \sigma \qquad a.e.\ in \quad I_- \cup I_+, \tag{3.12}$$

$$|\det J_{\Phi,t_o}| \ge \eta \qquad a.e.\ in \quad I. \tag{3.13}$$

If $h = \omega/\ell$ the intervals I_- and I_+ are empty. In this case E_{Φ,t_o} is a Riesz basis for B_ω if and only if conditions (3.11) and (3.13) hold.

Proof. First we shall prove the theorem for $\omega/\ell < h < \omega/(\ell - 1/2)$.

Let E_{Φ,t_o} be a frame for B_ω with frame constants A and B. This implies in particular that B_ω coincides with the space S_{Φ,t_o} generated by E_{Φ,t_o}. Condition (3.11) follows from Proposition 3.2. Thus we only need to prove (3.12) and (3.13).

We recall that the columns of the pre-Gramian J_{Φ,t_o} are the vectors $\mathcal{L}_h \widehat{\varphi_j}$, $j = 1, \dots, 2\ell$ by (2.14). Thus, by Theorem 3.4(i), the columns of $J_{\Phi,t_o}(x)$ are a frame with constants A, B for the space $\mathcal{R}_h(B(\omega))(x)$ for a.e. x. By Theorem 2.2 we may identify canonically $\mathcal{R}_h(B(\omega))(x)$ with $\mathbb{C}^{2\ell-1}$ for a.e. $x \in I_- \cup I_+$ and with $\mathbb{C}^{2\ell}$ for a.e. x in I. Thus, by applying Lemma 3.5(i) with $v_j = \mathcal{L}_h \widehat{\varphi_j}(x)$ and $J = J_{\Phi,t_o}(x)$, we obtain that $[J_{\Phi,t_o}(x)]_{2\ell-1} \ge A^{2\ell-1}$ for a.e. x in $I_- \cup I_+$ and $\det(J_{\Phi,t_o}(x)) = [J_{\Phi,t_o}(x)]_{2\ell}^{1/2} \ge A^\ell$ for a.e. x in I. This proves that conditions (3.11)-(3.13) are necessary.

To prove sufficiency assume that conditions (3.11)-(3.13) are satisfied. First we prove that the space S_{Φ,t_o} spanned by E_{Φ,t_o} is B_ω. Since both are t_o-shift invariant spaces it is enough to show that their range functions coincide almost everywhere. We recall that the range $\mathcal{R}_h(S_{\Phi,t_o})$ of S_{Φ,t_o} is the space spanned by the columns of J_{Φ,t_o}. If x is in I_- then by (3.9) $\mathcal{R}_h(S_{\Phi,t_o})(x) \subseteq \operatorname{span}\{\delta_j : |j| \leq \ell-1\}$ and the latter space coincides with $\mathcal{R}_h(B_\omega)(x)$ by (2.22). On the other hand, rank $J_{\Phi,t_o}(x) = 2\ell-1$ by (3.9) and assumption (3.12). Thus $\mathcal{R}_h(S_{\Phi,t_o})(x) = \mathcal{R}_h(B_\omega)(x)$ because both have dimension $2\ell-1$. Similar arguments show that the range functions coincide almost everywhere also in I_+ and in I.

Next we observe that $\|J_{\Phi,t_o}(x)\| \leq \sqrt{2\ell\gamma}$ for a.e. x in $[0,h]$ by (3.11). Moreover $[J_{\Phi,t_o}(x)]_{2\ell-1} > 0$ for a.e. x in $I_- \cup I_+$ by (3.12) and $[J_{\Phi,t_o}(x)]_{2\ell} = |\det(J_{\Phi,t_o})|^2 > 0$ for a.e. x in I by (3.13). Thus, by Lemma 3.5(i), the family $\{\mathcal{L}_h\widehat{\varphi_1}(x), \ldots, \mathcal{L}_h\widehat{\varphi_{2\ell}}(x)\}$ is a frame for $\mathcal{R}_h(B_\omega)(x)$ for a.e. x in $[0,h]$. The upper frame constant $B = \|J_{\Phi,t_o}(x)\|^2$ is bounded from above by $2\ell\gamma$ almost everywhere in $[0,h]$. The lower frame constant A is bounded from below by

$$[J_{\Phi,t_o}(x)]_{2\ell-1}\, \|J_{\Phi,t_o}(x)\|^{2(1-2\ell)} \geq \sigma\, (2\ell\gamma)^{(1-2\ell)} \quad \text{a.e. in } I_- \cup I_+$$

and by

$$|\det J_{\Phi,t_o}(x)|^2\, \|J_{\Phi,t_o}(x)\|^{2(1-2\ell)} \geq \eta^2\, (2\ell\gamma)^{(1-2\ell)} \quad \text{a.e. in } I.$$

Thus E_{Φ,t_o} is a frame for B_ω by Theorem 3.4(i). This concludes the proof of the theorem when $\omega/\ell < h < \omega/(\ell-1/2)$.

To prove that if $h = \omega/\ell$ then E_{Φ,t_o} is a Riesz basis for B_ω if and only if conditions (3.11) and (3.13) hold, one argues in a similar way using Theorem 3.4(ii) and Lemma 3.5(ii). We omit the details. □

Remark If $\ell = 1$ condition (3.12) is superfluous. Indeed, if $h = \omega$ the intervals I_- and I_+ are empty. If $\omega < h < 2\omega$ then (3.12) follows from (3.11) because

$$[J_{\Phi,t_o}]_{2\ell-1} = \sum_{j=1}^{2} [\widehat{\varphi_j}, \widehat{\varphi_j}] = \sum_k \tau_{kh} \sum_{j=1}^{2} |\widehat{\varphi_j}|^2 = \begin{cases} \sum_{j=1}^{2} |\widehat{\varphi_j}|^2 & \text{in } I_- \\ \\ \tau_{-h} \sum_{j=1}^{2} |\widehat{\varphi_j}|^2 & \text{in } I_+, \end{cases}$$

and the conclusion follows because $I_- \subset (-\omega,\omega)$ and $I_+ \subset \tau_h(-\omega,\omega)$.

Next we consider the case $\omega/(\ell-1/2) \leq h < \omega/(\ell-1)$. Then $\operatorname{len}_{t_o}(B_\omega) = 2\ell-1$ by Corollary 2.3. Let $\Phi = \{\varphi_j : 1 \leq j \leq 2\ell-1\}$ be a subset of B_ω of cardinality $2\ell-1$. By Lemma 3.3 all the rows of the matrix J_{Φ,t_o}, except possibly $\big(\tau_{jh}\widehat{\varphi}_1, \tau_{jh}\widehat{\varphi}_2, \ldots, \tau_{jh}\widehat{\varphi}_{2\ell}\big)$, $-\ell \leq j \leq \ell-1$ vanish. Thus we identify the infinite matrices J_{Φ,t_o}, J^*_{Φ,t_o} and G_{Φ,t_o} with their $2\ell-1 \times 2\ell-1$ submatrices corresponding to these rows. The i-th column of J_{Φ,t_o}, $1 \leq i \leq 2\ell-1$ is

(3.14)

$$\sqrt{h}\begin{bmatrix} \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ \tau_{(\ell-1)h}\widehat{\varphi}_i \end{bmatrix} \text{ in } K_- \qquad \sqrt{h}\begin{bmatrix} \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \\ 0 \end{bmatrix} \text{ in } K \qquad \sqrt{h}\begin{bmatrix} \tau_{-\ell h}\widehat{\varphi}_i \\ \tau_{-(\ell-1)h}\widehat{\varphi}_i \\ \vdots \\ \widehat{\varphi}_i \\ \\ \vdots \\ \tau_{(\ell-2)h}\widehat{\varphi}_i \end{bmatrix} \text{ in } K_+.$$

Theorem 3.7. *Let $\Phi \subset B_\omega$ such that $\Phi = \{\varphi_j, 1 \le j < 2\ell - 1\}$, and $\ell \ne 1$ such that $\omega/(\ell - 1/2) \le h < \omega/(\ell - 1)$. Then E_{Φ,t_o} is a frame for B_ω if and only if there exist positive constants δ, γ, σ and η such that*

$$\delta \le \sum_{j=1}^{2\ell-1} |\widehat{\varphi}_j|^2 \le \gamma \qquad \text{a.e. in } (-\omega, \omega), \tag{3.15}$$

$$[\, J_{\Phi,t_o}]_{2\ell-2} \ge \sigma \qquad \text{a.e. in } K, \tag{3.16}$$

$$|\det J_{\Phi,t_o}| \ge \eta \qquad \text{a.e. in } K_- \cup K_+. \tag{3.17}$$

If $h = \omega/(\ell - 1/2)$ then E_{Φ,t_o} is a Riesz basis if and only if conditions (3.15) and (3.17) hold.

The proof is similar to that of Theorem 3.6. We omit the details.

4. The dual generators

Let $N = \operatorname{len}_{t_o}(B_\omega)$ be the length of B_ω as t_o-shift-invariant space and let $\Phi = \{\varphi_1, \dots, \varphi_N\}$ be a subset of B_ω. In this section we shall find the dual generators Φ^* when E_{Φ,t_o} is a Riesz basis or a frame of B_ω. With a slight abuse of notation in this section we shall denote by Φ the vector $(\varphi_1, \dots, \varphi_N)$ and by Φ^* the vector $(\varphi_1^*, \dots \varphi_N^*)$.

It is well known that if E_{Φ,t_o} is a Riesz basis for S_{Φ,t_o} then the Gramian matrix is invertible and the Fourier transform of the dual generators are given by

$$\widehat{\Phi^*}^\top = \overline{G_{\Phi,t_o}^{-1}} \widehat{\Phi}^\top, \tag{4.1}$$

where $v^\top$ denotes the transpose of the vector v. In Theorems 4.1 - 4.7 we give explicit formulas for the Fourier transforms of the dual generators when E_{Φ,t_o} is a frame satisfying the hypothesis of Theorems 3.6 or 3.7. The proof is based on the dual Gramian matrix $\widetilde{G}_{\Phi,t_o}$ representation of the operator $T_{\Phi,t_o} T^*_{\Phi,t_o}$. From (3.4) we obtain

$$\mathcal{L}_h \mathcal{F} T_{\Phi,t_o} T^*_{\Phi,t_o} \mathcal{F}^{-1} \widehat{\varphi_k}^* = J_{\Phi,t_o} J^*_{\Phi,t_o}\, \mathcal{L}_h \widehat{\varphi_k}^* \qquad k = 1, \dots, N.$$

By (2.4) the left hand side is equal to $\mathcal{L}_h \widehat{\varphi_k}$. Hence

$$\mathcal{L}_h \widehat{\varphi_k} = J_{\Phi,t_o} J^*_{\Phi,t_o}\, \mathcal{L}_h \widehat{\varphi_k^*}, \qquad k = 1, \dots .N$$

which, by (2.14), can be written

$$J_{\Phi,t_o} = J_{\Phi,t_o} J^*_{\Phi,t_o}\, J_{\Phi^*,t_o}. \tag{4.2}$$

As in Section 2 we identify the infinite matrices J_{Φ,t_o}, J^*_{Φ,t_o} and J_{Φ^*,t_o} with $N \times N$ matrices by neglecting their vanishing rows and columns (see the discussion after Lemma 3.3). Thus we shall interpret (4.2) as an identity between $N \times N$ matrices.

Under the assumptions of Theorems 3.6 and 3.7 the interval $[0, h]$ is the disjoint union of three intervals where the pre-Gramian is either invertible or has rank $N-1$. In the latter case either the first or the last row of the pre-Gramian vanishes. In this case we shall denote by $\mathbb{J}_{\Phi,t_o}$ the $(N-1) \times N$ submatrix of J_{Φ,t_o} obtained by deleting the vanishing row from J_{Φ,t_o}. It is straightforward to see that in this case equation (4.2) reduces to

$$\mathbb{J}_{\Phi,t_o} = \mathbb{J}_{\Phi,t_o} \mathbb{J}^*_{\Phi,t_o}\, \mathbb{J}_{\Phi^*,t_o}. \tag{4.3}$$

We regard (4.2) and (4.3) as equations for the unknowns J_{Φ^*,t_o} and $\mathbb{J}_{\Phi^*,t_o}$, respectively. In the intervals where the matrix J_{Φ,t_o} is invertible we can solve for J_{Φ^*,t_o} in (4.2), obtaining that

$$J_{\Phi^*,t_o} = (J^*_{\Phi,t_o})^{-1}. \tag{4.4}$$

In the intervals where the rank of J_{Φ,t_o} is $N-1$ we can solve for the submatrix $\mathbb{J}_{\Phi^*,t_o}$ obtaining that

$$\mathbb{J}_{\Phi^*,t_o} = (\mathbb{J}_{\Phi,t_o}\mathbb{J}^*_{\Phi,t_o})^{-1}\mathbb{J}_{\Phi,t_o}. \tag{4.5}$$

We recall that if A is a $N \times (N-1)$ matrix of rank $N-1$ then its Moore–Penrose inverse $A^\dagger$ is

$$A^\dagger = (A^*A)^{-1}A^* \tag{4.6}$$

(see [BIG]). Therefore by (4.5)

$$\mathbb{J}_{\Phi^*,t_o} = (\mathbb{J}^*_{\Phi,t_o})^\dagger. \tag{4.7}$$

We refer the reader to [BIG] for the definition and the properties of the Moore–Penrose inverse of a matrix.

By using (4.4) and (4.5) we shall obtain explicit formulas for the Fourier transforms of the dual generators. For the sake of clarity first we state and prove the result for $N = 2, 3, 4$. By Corollary 2.3 these cases correspond to $\omega \leq h < 2\omega$, $\omega/2 \leq h < 2\omega/3$ and $2\omega/3 \leq h < \omega$ respectively.

Theorem 4.1. *Assume that $\omega \leq h < 2\omega$ and let Φ denote the vector (φ_1, φ_2) where $\varphi_1, \varphi_2 \in B_\omega$. If (3.11) and (3.13) hold with $\ell = 1$, i.e. if E_{Φ,t_o} is a frame for B_ω, then*

$$\widehat{\Phi^*} = \begin{cases} D\,\tau_h\,\overline{\widehat{\Phi}}^{\perp} & \text{in } [-\omega, \omega-h] \\ h^{-1}\|\widehat{\Phi}\|^{-2}\ \widehat{\Phi} & \text{in } (\omega-h, h-\omega) \\ -D\,\tau_{-h}\,\overline{\widehat{\Phi}}^{\perp} & \text{in } [h-\omega, \omega] \end{cases} \tag{4.8}$$

where $D = (\det\ J^*_{\Phi,t_o})^{-1}$ and $\widehat{\Phi}^\perp = (\widehat{\varphi_2}, -\widehat{\varphi_1})$. Note that if $h = \omega$ the central interval is empty.

Proof. Assume first that $\omega < h < 2\omega$. We recall that $I_- = (0, h-\omega)$, $I = (h-\omega, \omega)$ and $I_+ = (\omega, h)$. The pre-Gramian J_{Φ,t_o} is

$$\sqrt{h}\begin{bmatrix} 0 & 0 \\ \widehat{\varphi_1} & \widehat{\varphi_2} \end{bmatrix} \text{in } \ I_- \qquad \sqrt{h}\begin{bmatrix} \tau_{-h}\widehat{\varphi_1} & \tau_{-h}\widehat{\varphi_2} \\ 0 & 0 \end{bmatrix} \text{ in } \ I_+$$

$$\sqrt{h}\begin{bmatrix} \tau_{-h}\widehat{\varphi_1} & \tau_{-h}\widehat{\varphi_2} \\ \widehat{\varphi_1} & \widehat{\varphi_2} \end{bmatrix} \text{in } \ I.$$

The same formulas hold for J_{Φ^*,t_o} with $\widehat{\varphi}_i$ replaced by $\widehat{\varphi_i^*}$, $j = 1, 2$. Therefore

$$\mathbb{J}_{\Phi,t_o} = \begin{cases} \sqrt{h}\,\widehat{\Phi} & \text{in } I_- \\ \sqrt{h}\,\tau_{-h}\widehat{\Phi} & \text{in } I_+ \end{cases} \qquad \mathbb{J}_{\Phi^*,t_o} = \begin{cases} \sqrt{h}\,\widehat{\Phi^*} & \text{in } I_- \\ \sqrt{h}\,\tau_{-h}\widehat{\Phi^*} & \text{in } I_+\,. \end{cases} \tag{4.9}$$

By assumptions (3.11) and (3.13) J_{Φ,t_o} has rank 1 in $I_- \cup I_+$ and rank 2 in I. Hence $\mathbb{J}_{\Phi^*,t_o} = (\mathbb{J}^*_{\Phi,t_o})^\dagger$ in $I_- \cup I_+$ and

$$J_{\Phi^*,t_o} = {J^*_{\Phi,t_o}}^{-1} \qquad \text{in } \ I. \tag{4.10}$$

First we find $\mathbb{J}_{\Phi^*,t_o}$ in $I_- \cup I_+$. By using (4.5) we get

$$\mathbb{J}_{\Phi^*,t_o} = (\mathbb{J}^*_{\Phi,t_o})^\dagger = \begin{cases} \frac{\widehat{\Phi}}{\sqrt{h}\|\widehat{\Phi}\|^2} & \text{in } I_- \\ \\ \frac{\tau_{-h}\widehat{\Phi}}{\sqrt{h}\|\tau_{-h}\widehat{\Phi}\|^2} & \text{in } I_+. \end{cases}$$

By (4.9) we obtain that $\widehat{\Phi^*} = \frac{\widehat{\Phi}}{h\|\widehat{\Phi}\|^2}$ in I_- and $\tau_{-h}\widehat{\Phi^*} = \frac{\tau_{-h}\widehat{\Phi}}{h\|\tau_{-h}\widehat{\Phi}\|^2}$ in I_+. Since $\tau_{-h}I_+ = (\omega - h, 0)$ we finally get

$$\widehat{\Phi^*} = \frac{\widehat{\Phi}}{h\|\widehat{\Phi}\|^2} \qquad \text{in } (\omega - h, h - \omega).$$

Next we find the dual generators in the remaining intervals. By (4.10)

$$J_{\Phi^*,t_o} = \sqrt{h}\ (\det J^*_{\Phi,t_o})^{-1} \begin{bmatrix} \overline{\widehat{\varphi_2}} & -\overline{\widehat{\varphi_1}} \\ -\tau_{-h}\overline{\widehat{\varphi_2}} & \tau_{-h}\overline{\widehat{\varphi_1}} \end{bmatrix} \qquad \text{a.e. in} \quad (h-\omega, \omega).$$

By translating and reminding that the pre-Gramian matrix is h-periodic

$$\widehat{\Phi}^* = \begin{cases} D\,\tau_h\big(\overline{\widehat{\varphi_2}}, -\overline{\widehat{\varphi_1}}\big) = D\,\tau_h\,\overline{\widehat{\Phi}}^{\perp} & \text{in} \quad (-\omega, \omega - h) \\ D\,\tau_{-h}\big(-\overline{\widehat{\varphi_2}}, \overline{\widehat{\varphi_1}}\big) = -D\,\tau_{-h}\,\overline{\widehat{\Phi}}^{\perp} & \text{in} \quad (h-\omega, \omega). \end{cases}$$

This completes the proof of the theorem when $\omega < h < 2\omega$.

If $h = \omega$ one argues as before; the only difference is that now the interval $(h - \omega, \omega - h)$ is empty. □

To find an explicit expression of the dual generators when $N > 2$ we need formulas for the rows of the Moore–Penrose inverse of a $N \times (N-1)$ matrix of full rank. Given a $(N-1)$-ple of vectors $(U_1, \ldots, U_{N-1})$ in $\mathbb{C}^N$ their cross product is

$$\bigtimes_{1}^{N-1} U_j = U_1 \times U_2 \ldots \times U_{N-1} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_N \\ U_1^1 & U_1^2 & \ldots & U_1^N \\ \vdots & \vdots & & \vdots \\ U_{N-1}^1 & U_{N-1}^2 & \ldots & U_{N-1}^N \end{bmatrix}$$

where the $\{\mathbf{e_j} : j = 1, \ldots, N\}$ is the canonical basis of $\mathbb{C}^N$. Notice that if $N = 2$ then $\bigtimes U = U^\perp$. Given a vector W in $\mathbb{C}^N$ and an integer $k \in \{1, 2, \ldots, N-1\}$, we shall denote by $\bigtimes_{j=1}^{N-1} U_j \left\langle U_k \leftarrow W \right\rangle$ the cross product of the $(N-1)$-ple $(U_1, \ldots, U_{k-1}, W, U_{k+1}, \ldots, U_{N-1})$, i.e.

$$\bigtimes_{j=1}^{N-1} U_j \left\langle U_k \leftarrow W \right\rangle = U_1 \times U_2 \ldots \times U_{k-1} \times W \times U_{k+1} \ldots \times U_{N-1}.$$

Lemma 4.2. *Le M be an $n \times n$ invertible matrix. Denote by R_j, $j = 1, \ldots, n$, its rows and by C_j its columns. Then the columns of M^{-1} are*

$$(-1)^{k+1}(\det M)^{-1} \bigtimes_{\substack{j=1 \\ j \neq k}}^{n} R_j \qquad 1 \le k \le n$$

and the rows are

$$(-1)^{k+1}(\det M)^{-1} \bigtimes_{\substack{j=1\\ j\neq k}}^{n} C_j \qquad 1 \le k \le n.$$

Proof. Let M_{kj} be the kj-cofactor of M. Then the k-th row of the matrix $\operatorname{cof}(M)$ of cofactors of M is

$$\begin{aligned} L_k &= (-1)^{k+1}(M_{k1}\mathbf{e}_1 - M_{k2}\mathbf{e}_2 + \cdots + (-1)^{n-1}M_{kn}\mathbf{e}_n) \\ &= (-1)^{k+1} \bigtimes_{\substack{j=1\\ j\neq k}}^{n} R_j. \end{aligned}$$

The first identity follows from the fact that $M^{-1} = (\det M)^{-1}\operatorname{cof}(M^\top)$. The proof of the second identity is similar. □

Lemma 4.3. *Le A be a $n \times (n-1)$ complex matrix of maximum rank. Denote by A_j, $j = 1, \dots, n-1$, the columns of A and by P_k, $k = 1, \dots, n-1$ the rows of $A^\dagger$. Then*

$$P_k = (-1)^n [\det(A^*A)]^{-1} \bigtimes_{j=1}^{n-1} A_j \left\langle\, A_k \leftarrow W \,\right\rangle \qquad 1 \le k \le n-1 \tag{4.11}$$

where $W = \bigtimes_{j=1}^{n-1} \overline{A}_j$.

Proof. Since $\operatorname{rank} A = n-1$ the null space of A^* is the space spanned by W. Let A_b the matrix obtained by bordering A with the column W, i.e.

$$A_b = \begin{bmatrix} A & W \end{bmatrix}.$$

Then A_b is invertible because $\det A_b = \bigtimes_{j=1}^{n-1} A_j \cdot W = |W|^2$. By [BIG, Thm. 8]

$$A_b^{-1} = \begin{bmatrix} A^\dagger \\ W^\dagger \end{bmatrix}.$$

Thus, for every $k = 1, \dots, n-1$, the k-th row of $A^\dagger$ is the k-th row of A_b^{-1}. Hence, by Lemma 4.2 and the anticommutativity of the cross product, we obtain that

$$\begin{aligned} P_k =& (-1)^{k+1}\big(\det A_b\big)^{-1} A_1 \times A_2 \times \ldots \times A_{k-1} \times A_{k+1} \ldots \times A_{n-1} \times W \\ =& (-1)^n |W|^{-2} \bigtimes_{j=1}^{n-1} A_j \left\langle\, A_k \leftarrow W \,\right\rangle. \end{aligned}$$

To conclude the proof we observe that $|W|^2$ is the sum of the squares of the absolute values of the minors of order $n-1$ of A, i.e. the determinant of A^*A, by the Cauchy-Binet formula. □

Theorem 4.4. *Assume that $\frac{2}{3}\omega \le h < \omega$ and let Φ denote the vector $(\varphi_1, \varphi_2, \varphi_3)$ where $\varphi_j \in B_\omega$, $j = 1, 2, 3$. If assumptions (3.15)-(3.17) hold with $\ell = 2$, i.e.*

if E_{Φ,t_o} is a frame for B_ω, then the Fourier transform of the dual generators $\Phi^* = (\varphi_1^*, \varphi_2^*, \varphi_3^*)$ is

$$\widehat{\Phi}^* = \begin{cases} D \quad \tau_h \overline{\widehat{\Phi}} \times \tau_{2h} \overline{\widehat{\Phi}} & in \quad (-\omega, \omega - 2h) \\ E \, \big(\tau_h \widehat{\Phi} \times \widehat{\Phi}\big) \times \tau_h \overline{\widehat{\Phi}} & in \quad (\omega - 2h, h - \omega) \\ D \quad \tau_h \overline{\widehat{\Phi}} \times \tau_{-h} \overline{\widehat{\Phi}} & in \quad (h - \omega, \omega - h) \\ -E \ \tau_{-h} \overline{\widehat{\Phi}} \times \big(\tau_{-h} \widehat{\Phi} \times \widehat{\Phi}\big) & in \quad (\omega - h, 2h - \omega) \\ D \quad \tau_{-2h} \overline{\widehat{\Phi}} \times \tau_{-h} \overline{\widehat{\Phi}} & in \quad (2h - \omega, \omega) \end{cases} \tag{4.12}$$

where $E = h \ (\det \mathbb{J}_{\Phi,t_o} \mathbb{J}^*_{\Phi,t_o})^{-1}$ and $D = \sqrt{h}\,(\det J^*_{\Phi,t_o})^{-1}$. Note that if $h = 2\omega/3$ the intervals $(\omega - h, 2h - \omega)$ $(\omega - 2h, h - \omega)$ are empty.

Proof. Assume first that $2\omega/3 < h < \omega$. We recall that K_-, K and K_+ denote the intervals $(0, \omega - h)$, $(\omega - h, 2h - \omega)$ and $(2h - \omega, h)$ respectively, defined in (2.21). By (3.14) the matrix J_{Φ,t_o} is

$$\sqrt{h} \begin{bmatrix} \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \\ \tau_h \widehat{\Phi} \end{bmatrix} \text{ in } K_- \qquad \sqrt{h} \begin{bmatrix} \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \\ 0 \end{bmatrix} \text{ in } K \qquad \sqrt{h} \begin{bmatrix} \tau_{-2h}\widehat{\Phi} \\ \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \end{bmatrix} \text{ in } K_+. \tag{4.13}$$

The same formulas hold for J_{Φ^*,t_o} with $\widehat{\Phi}$ replaced by $\widehat{\Phi^*}$. Therefore

$$\mathbb{J}_{\Phi,t_o} = \sqrt{h} \begin{bmatrix} \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \end{bmatrix} \quad \text{and} \quad \mathbb{J}_{\Phi^*,t_o} = \sqrt{h} \begin{bmatrix} \tau_{-h}\widehat{\Phi^*} \\ \widehat{\Phi^*} \end{bmatrix} \text{ in } K. \tag{4.14}$$

By assumptions (3.16) and (3.17) the matrix J_{Φ,t_o} has rank 3 in $K_- \cup K_+$ and rank 2 in K. Hence $J_{\Phi^*,t_o} = {J^*_{\Phi,t_o}}^{-1}$ in $K_- \cup K_+$ and $\mathbb{J}_{\Phi^*,t_o} = (\mathbb{J}^*_{\Phi,t_o})^\dagger$ in K.

First we find J_{Φ^*,t_o} in K_-. By Lemma 4.2

$$\tau_{-h}\widehat{\Phi^*} = D \, \overline{\widehat{\Phi}} \times \tau_h \overline{\widehat{\Phi}}, \qquad \widehat{\Phi^*} = D \, \tau_h \overline{\widehat{\Phi}} \times \tau_{-h} \overline{\widehat{\Phi}}, \qquad \tau_h \widehat{\Phi^*} = D \, \tau_{-h} \overline{\widehat{\Phi}} \times \overline{\widehat{\Phi}} \quad \text{a.e. in } K_-$$

where $D = \sqrt{h}\,(\det J^*_{\Phi,t_o})^{-1}$. By translating the first and the last identities and reminding that the pre-Gramian is h-periodic, we obtain

$$\widehat{\Phi^*} = \begin{cases} D \quad \tau_h \overline{\widehat{\Phi}} \times \tau_{2h} \overline{\widehat{\Phi}} & \text{in } (-h, \omega - 2h) \\ D \quad \tau_h \overline{\widehat{\Phi}} \times \tau_{-h} \overline{\widehat{\Phi}} & \text{in } (0, \omega - h) \\ D \ \tau_{-2h} \overline{\widehat{\Phi}} \times \tau_{-h} \overline{\widehat{\Phi}} & \text{in } (h, \omega). \end{cases} \tag{4.15}$$

The same calculation in K_+ gives

$$\tau_{-2h}\widehat{\Phi^*} = D \, \tau_{-h} \overline{\widehat{\Phi}} \times \overline{\widehat{\Phi}} \qquad \tau_{-h}\widehat{\Phi^*} = D \, \overline{\widehat{\Phi}} \times \tau_{-2h} \overline{\widehat{\Phi}} \qquad \widehat{\Phi^*} = D \, \tau_{-2h} \overline{\widehat{\Phi}} \times \tau_{-h} \overline{\widehat{\Phi}}.$$

By translating the first two identities we obtain

$$\widehat{\Phi^*} = \begin{cases} D \quad \tau_h \overline{\widehat{\Phi}} \times \tau_{2h} \overline{\widehat{\Phi}} & \text{in } (-\omega, -h) \\ D \quad \tau_h \overline{\widehat{\Phi}} \times \tau_{-h} \overline{\widehat{\Phi}} & \text{in } (h - \omega, 0) \\ D \ \tau_{-2h} \overline{\widehat{\Phi}} \times \tau_{-h} \overline{\widehat{\Phi}} & \text{in } (2h - \omega, h). \end{cases} \tag{4.16}$$

Next we find $\mathbb{J}_{\Phi^*,t_o}$ in K. By (4.14) and Lemma 4.3 the rows of Moore–Penrose inverse of $\mathbb{J}^*_{\Phi,t_o}$ are

$$-\sqrt{h}E \, \big(\tau_{-h}\widehat{\Phi} \times \widehat{\Phi}\big) \times \overline{\widehat{\Phi}} \qquad\qquad -\sqrt{h}E \ \tau_{-h} \overline{\widehat{\Phi}} \times \big(\tau_{-h}\widehat{\Phi} \times \widehat{\Phi}\big)$$

where $E = h\ (\det \mathbb{J}_{\Phi,t_o}\mathbb{J}^*_{\Phi,t_o})^{-1}$. Hence, by (4.14),

$$\tau_{-h}\widehat{\Phi^*} = -E\,\big(\tau_{-h}\widehat{\Phi} \times \widehat{\Phi}\big) \times \overline{\widehat{\Phi}} \qquad\qquad \widehat{\Phi^*} = -E\ \tau_{-h}\overline{\widehat{\Phi}} \times \big(\tau_{-h}\widehat{\Phi} \times \widehat{\Phi}\big).$$

By translating the first identity and using the anticommutativity of the cross product, we obtain

$$\widehat{\Phi^*} = \begin{cases} E\ \tau_h\overline{\widehat{\Phi}} \times \big(\widehat{\Phi} \times \tau_h\widehat{\Phi}\big) & \text{in } (\omega-2h, h-\omega) \\ E\ \big(\tau_{-h}\widehat{\Phi} \times \widehat{\Phi}\big) \times \tau_{-h}\overline{\widehat{\Phi}} & \text{in } (\omega-h, 2h-\omega). \end{cases} \tag{4.17}$$

The conclusion follows from formulas (4.15), (4.16) and (4.17). This completes the proof of the theorem when $2\omega/3 < h < \omega$.

If $h = 2\omega/3$ one argues as before; the only difference is that now the interval $(\omega-h, 2h-\omega)$ is empty. □

Theorem 4.5. *Assume that $\omega/2 \leq h < 2\omega/3$ and let Φ denote the vector $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ where $\varphi_j \in B_\omega$, $j = 1, \ldots, 4$. If (3.11)-(3.13) hold with $\ell = 2$, i.e. if E_{Φ,t_o} is a frame for B_ω, then the Fourier transform of the dual generators $\Phi^* = (\varphi_1^*, \varphi_2^*, \varphi_3^*, \varphi_4^*)$ is*

$$\widehat{\Phi}^* = \begin{cases} E & \tau_{-h}\overline{\widehat{\Phi}} \times \big(\tau_{-h}\widehat{\Phi} \times \widehat{\Phi} \times \tau_h\widehat{\Phi}\big) \times \tau_h\overline{\widehat{\Phi}} & \text{in } (0, 2h-\omega) \\ D & \tau_{-2h}\overline{\widehat{\Phi}} \times \tau_{-h}\overline{\widehat{\Phi}} \times \tau_h\overline{\widehat{\Phi}} & \text{in } (2h-\omega, \omega-h) \\ E & \tau_{-2h}\overline{\widehat{\Phi}} \times \tau_{-h}\overline{\widehat{\Phi}} \times \big(\tau_{-2h}\widehat{\Phi} \times \tau_{-h}\widehat{\Phi} \times \widehat{\Phi}\big) & \text{in } (\omega-h, 3h-\omega) \\ D & \tau_{-3h}\overline{\widehat{\Phi}} \times \tau_{-2h}\overline{\widehat{\Phi}} \times \tau_{-h}\overline{\widehat{\Phi}} & \text{in } (3h-\omega, \omega) \end{cases} \tag{4.18}$$

where $E = h^2\ (\det \mathbb{J}_{\Phi,t_o}\mathbb{J}^*_{\Phi,t_o})^{-1}$ and $D = h\,(\det\overline{J_{\Phi,t_o}})^{-1}$. The expression of Φ^* in $(-\omega, 0)$ is obtained by reflecting each interval in (4.18) around zero and replacing τ_{jh} with τ_{-jh} in the expression of $\widehat{\Phi}^*$ in the reflected interval. Note that if $h = \omega/2$ then the first and third intervals are empty.

Proof. Assume first that $\omega/2 < h < 2\omega/3$. We recall that by I_-, I and I_+ we denote the intervals $(0, 2h-\omega)$, $(2h-\omega, \omega-h)$ and $(\omega-h, h)$ defined in (2.20). By (3.9) the columns of J_{Φ,t_o} are

$$\sqrt{h}\begin{bmatrix} 0 \\ \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \\ \tau_h\widehat{\Phi} \end{bmatrix} \text{ in } I_- \qquad \sqrt{h}\begin{bmatrix} \tau_{-2h}\widehat{\Phi} \\ \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \\ \tau_h\widehat{\Phi} \end{bmatrix} \text{ in } I \qquad \sqrt{h}\begin{bmatrix} \tau_{-2h}\widehat{\Phi} \\ \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \\ 0 \end{bmatrix} \text{ in } I_+.$$

The same formulas hold for J_{Φ^*,t_o} with $\widehat{\Phi}$ replaced by $\widehat{\Phi^*}$. Therefore

$$\mathbb{J}_{\Phi,t_o} = \sqrt{h}\begin{bmatrix} \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \\ \tau_h\widehat{\Phi} \end{bmatrix} \qquad \text{and} \qquad \mathbb{J}_{\Phi^*,t_o} = \sqrt{h}\begin{bmatrix} \tau_{-h}\widehat{\Phi^*} \\ \widehat{\Phi^*} \\ \tau_h\widehat{\Phi^*} \end{bmatrix} \quad \text{in } I_- \tag{4.19}$$

$$\mathbb{J}_{\Phi,t_o} = \sqrt{h}\begin{bmatrix} \tau_{-2h}\widehat{\Phi} \\ \tau_{-h}\widehat{\Phi} \\ \widehat{\Phi} \end{bmatrix} \qquad \text{and} \qquad \mathbb{J}_{\Phi^*,t_o} = \sqrt{h}\begin{bmatrix} \tau_{-2h}\widehat{\Phi^*} \\ \tau_{-h}\widehat{\Phi^*} \\ \widehat{\Phi^*} \end{bmatrix} \quad \text{in } I_+. \tag{4.20}$$

By assumptions (3.12) and (3.13) the matrix J_{Φ,t_o} has rank 3 in $I_- \cup I_+$ and rank 4 in I. Hence $\mathbb{J}_{\Phi^*,t_o} = (\mathbb{J}^*_{\Phi,t_o})^\dagger$ in $I_- \cup I_+$ and $J_{\Phi^*,t_o} = {J^*_{\Phi,t_o}}^{-1}$ in I.

First we find J_{Φ^*,t_o} in I; by Lemma 4.2

$$
\begin{aligned}
\tau_{-2h}\widehat{\Phi^*} &= D \quad \tau_{-h}\,\overline{\widehat{\Phi}} \times \overline{\widehat{\Phi}} \times \tau_h\,\overline{\widehat{\Phi}} \\
\tau_{-h}\widehat{\Phi^*} &= -\,D \quad \tau_{-2h}\,\overline{\widehat{\Phi}} \times \overline{\widehat{\Phi}} \times \tau_h\,\overline{\widehat{\Phi}} \\
\widehat{\Phi^*} &= D \quad \tau_{-2h}\,\overline{\widehat{\Phi}} \times \tau_{-h}\,\overline{\widehat{\Phi}} \times \tau_h\,\overline{\widehat{\Phi}} \\
\tau_h\widehat{\Phi^*} &= -\,D \quad \tau_{-2h}\,\overline{\widehat{\Phi}} \times \tau_{-h}\,\overline{\widehat{\Phi}} \times \overline{\widehat{\Phi}}
\end{aligned}
$$

in $(2h-\omega,\omega-h)$; here $D = h\,(\det\overline{J_{\Phi,t_o}})^{-1}$. By translating and reminding that the matrix J_{Φ,t_o} is h-periodic we obtain

$$
\widehat{\Phi^*} = \begin{cases}
D \quad \tau_h\,\overline{\widehat{\Phi}} \times \tau_{2h}\,\overline{\widehat{\Phi}} \times \tau_{3h}\,\overline{\widehat{\Phi}} & \text{in} \quad (-\omega,\omega-3h) \\
-\,D \quad \tau_{-h}\,\overline{\widehat{\Phi}} \times \tau_h\,\overline{\widehat{\Phi}} \times \tau_{2h}\,\overline{\widehat{\Phi}} & \text{in} \quad (h-\omega,\omega-2h) \\
D \quad \tau_{-2h}\,\overline{\widehat{\Phi}} \times \tau_{-h}\,\overline{\widehat{\Phi}} \times \tau_h\,\overline{\widehat{\Phi}} & \text{in} \quad (2h-\omega,\omega-h) \\
-\,D \quad \tau_{-3h}\,\overline{\widehat{\Phi}} \times \tau_{-2h}\,\overline{\widehat{\Phi}} \times \tau_{-h}\,\overline{\widehat{\Phi}} & \text{in} \quad (3h-\omega,\omega).
\end{cases} \tag{4.21}
$$

Notice that if $[a,b]$ is any of the intervals in the r.h.s. of (4.21) the expression of $\widehat{\Phi^*}$ in $[a,b]$ can be obtained from that in $[-b,-a]$, by replacing h by $-h$ in the translations $\tau_{jh}, |j| \le 3$.

Next we find the dual generators in the remaining intervals. First let us consider the interval I_-. Here by Lemma 4.3 the rows of the Moore–Penrose inverse of $\mathbb{J}^*_{\Phi,t_o}$ are

$$
\sqrt{h}\,E\;W \times \overline{\widehat{\Phi}} \times \tau_h\,\overline{\widehat{\Phi}}, \qquad \sqrt{h}\,E\;\tau_{-h}\,\overline{\widehat{\Phi}} \times W \times \tau_h\,\overline{\widehat{\Phi}}, \qquad \sqrt{h}\,E\;\tau_{-h}\,\overline{\widehat{\Phi}} \times \overline{\widehat{\Phi}} \times W
$$

where $W = \tau_{-h}\widehat{\Phi} \times \widehat{\Phi} \times \tau_h\widehat{\Phi}$ and $E = h^2\ (\det\,\mathbb{J}_{\Phi,t_o}\mathbb{J}^*_{\Phi,t_o})^{-1}$. By using (4.19) and translating we obtain

$$
\widehat{\Phi^*} = \begin{cases}
E \quad \tau_h W \times \tau_h\widehat{\Phi} \times \tau_{2h}\,\overline{\widehat{\Phi}} & \text{in} \quad (-h,h-\omega) \\
E \quad \tau_{-h}\,\overline{\widehat{\Phi}} \times W \times \tau_h\,\overline{\widehat{\Phi}} & \text{in} \quad (0,2h-\omega) \\
E \quad \tau_{-2h}\,\overline{\widehat{\Phi}} \times \tau_{-h}\,\overline{\widehat{\Phi}} \times \tau_{-h}W & \text{in} \quad (h,3h-\omega).
\end{cases} \tag{4.22}
$$

Finally we consider the interval I_+. Here the rows of the Moore–Penrose inverse of $\mathbb{J}^*_{\Phi,t_o}$ are

$$
\sqrt{h}E\,W_o \times \tau_{-h}\,\overline{\widehat{\Phi}} \times \overline{\widehat{\Phi}}, \qquad \sqrt{h}E\,\tau_{-2h}\,\overline{\widehat{\Phi}} \times W_o \times \overline{\widehat{\Phi}}, \qquad \sqrt{h}E\,\tau_{-2h}\,\overline{\widehat{\Phi}} \times \tau_{-h}\,\overline{\widehat{\Phi}} \times W_o
$$

where $W_o = \tau_{-2h}\widehat{\Phi} \times \tau_{-h}\widehat{\Phi} \times \widehat{\Phi}$. By using (4.20) and translating we obtain

$$
\widehat{\Phi^*} = \begin{cases}
E \quad \tau_{2h}W_o \times \tau_h\,\overline{\widehat{\Phi}} \times \tau_{2h}\,\overline{\widehat{\Phi}} & \text{in} \quad (\omega-3h,-h) \\
E \quad \tau_{-h}\,\overline{\widehat{\Phi}} \times \tau_h W_o \times \tau_h\,\overline{\widehat{\Phi}} & \text{in} \quad (\omega-2h,0) \\
E \quad \tau_{-2h}\,\overline{\widehat{\Phi}} \times \tau_{-h}\,\overline{\widehat{\Phi}} \times W_o & \text{in} \quad (\omega-h,h).
\end{cases}
$$

Since $\tau_h W_o = W$ we obtain

$$
\widehat{\Phi^*} = \begin{cases}
E \quad \tau_h W \times \tau_h\,\overline{\widehat{\Phi}} \times \tau_{2h}\,\overline{\widehat{\Phi}} & \text{in} \quad (\omega-3h,-h) \\
E \quad \tau_{-h}\,\overline{\widehat{\Phi}} \times W \times \tau_h\,\overline{\widehat{\Phi}} & \text{in} \quad (\omega-2h,0) \\
E \quad \tau_{-2h}\,\overline{\widehat{\Phi}} \times \tau_{-h}\,\overline{\widehat{\Phi}} \times \tau_{-h}W & \text{in} \quad (\omega-h,h).
\end{cases} \tag{4.23}
$$

By comparing formulas (4.22) and (4.23) we see that the expressions of $\widehat{\Phi}^*$ in intervals symmetric with respect to zero can be obtained from each other by replacing τ_{jh} with τ_{-jh} (note that replacing τ_{jh} by τ_{-jh} changes also the sign of W).

Formulas (4.21), (4.22) and (4.23) give the dual generators. This completes the proof of the theorem when $\omega/2 < h < 2/3\ \omega$.

If $h = \omega/2$ one argues as before; the only difference is that now the intervals $(\omega - 2h, 2h - \omega)$, $(-h, h - \omega)$ and $(\omega - h, h)$ are empty. $\square$

Theorems 4.6 and 4.7 below generalize Theorems 4.4 and 4.5 respectively. We omit the proofs, which are analogous to the proofs of Theorems 4.4 and 4.5. We recall that K_-, K, and K_+ denote the intervals $(0, \omega-(\ell-1)h)$, $(\omega-(\ell-1)h, -\omega+\ell h)$, and $(-\omega + \ell h, h)$ respectively.

Theorem 4.6. *Let $\omega/(\ell - 1/2) \le h < \omega/(\ell - 1)$ and denote by Φ the vector $(\varphi_1, \varphi_2, \dots, \varphi_{2\ell-1})$, where $\varphi_j \in B_\omega$, $j = 1, \dots, 2\ell - 1$. If assumptions (3.15)-(3.17) hold, i.e. if E_{Φ,t_o} is a frame for B_ω, then the Fourier transform of the dual generators $\Phi^* = (\varphi_1^*, \varphi_2^*, \dots \varphi_{2\ell-1}^*)$ is*

$$\widehat{\Phi}^* = \begin{cases} (-1)^{\ell+1-k} D \displaystyle\bigtimes_{\substack{j=-\ell-k+1 \\ j\neq 0}}^{\ell-k-1} \tau_{jh} \overline{\widehat{\Phi}} & \text{in } \tau_{kh} K_- \quad \text{for } -(\ell-1) \le k \le \ell - 1 \\ -E \displaystyle\bigtimes_{j=-\ell-k+1}^{\ell-k-2} \tau_{jh} \overline{\widehat{\Phi}} \left\langle \overline{\widehat{\Phi}} \leftarrow \tau_{-kh} W \right\rangle & \text{in } \tau_{kh} K \quad \text{for } -(\ell-1) \le k \le \ell - 2 \\ (-1)^{\ell+k} D \displaystyle\bigtimes_{\substack{j=-\ell-k \\ j\neq 0}}^{\ell-2-k} \tau_{jh} \overline{\widehat{\Phi}} & \text{in } \tau_{kh} K_+ \quad \text{for } -\ell \le k \le \ell - 2 \end{cases}$$

where $W = \displaystyle\bigtimes_{j=-\ell+1}^{\ell-2} \tau_{jh} \widehat{\Phi}$, $E = h^{2\ell-3} (\det \mathbb{J}_{\Phi,t_o} \mathbb{J}^*_{\Phi,t_o})^{-1}$ *and* $D = h^{\ell-3/2} (\det J^*_{\Phi,t_o})^{-1}$.

We recall that I_-, I, I_+ denote the intervals $(0, -\omega + \ell h)$, $(-\omega + \ell h, \omega - (\ell - 1)h)$, and $(\omega - (\ell - 1)h, h)$ (see (2.20)).

Theorem 4.7. *Let $\omega/\ell \le h < \omega/(\ell - 1/2)$ and denote by Φ the vector $(\varphi_1, \varphi_2, \dots, \varphi_{2\ell})$ where $\varphi_j \in B_\omega$, $j = 1, \dots, 2\ell$. If assumptions (3.11)-(3.13) hold, i.e. if $E_{\Phi,t_o}$is a frame for B_ω, then the Fourier transform of the dual generators $\Phi^* = (\varphi_1^*, \varphi_2^*, \dots \varphi_{2\ell}^*)$ is*

$$\widehat{\Phi}^* = \begin{cases} E \displaystyle\bigtimes_{j=-\ell-k+1}^{\ell-k-1} \tau_{jh} \overline{\widehat{\Phi}} \left\langle \overline{\widehat{\Phi}} \leftarrow \tau_{-kh} W \right\rangle & \text{in } \tau_{kh} I_- \quad \text{for } -(\ell-1) \le k \le \ell - 1 \\ (-1)^{\ell+k} D \displaystyle\bigtimes_{\substack{j=-\ell-k \\ j\neq 0}}^{\ell-k-1} \tau_{jh} \overline{\widehat{\Phi}} & \text{in } \tau_{kh} I \quad \text{for } -\ell \le k \le \ell - 1 \\ E \displaystyle\bigtimes_{j=-\ell-k}^{\ell-k-2} \tau_{jh} \overline{\widehat{\Phi}} \left\langle \overline{\widehat{\Phi}} \leftarrow \tau_{-kh} W_o \right\rangle & \text{in } \tau_{kh} I_+ \quad \text{for } -\ell \le k \le \ell - 2 \end{cases}$$

where $W = \bigtimes_{j=-\ell+1}^{\ell-1} \tau_{jh}\widehat{\Phi}$, $W_o = \bigtimes_{j=-\ell}^{\ell-2} \tau_{jh}\widehat{\Phi}$, $E = h^{2\ell-2}\,(\det\, \mathbb{J}_{\Phi,t_o}\mathbb{J}^*_{\Phi,t_o})^{-1}$ *and* $D = h^{\ell-1}\,(\det\, J^*_{\Phi,t_o})^{-1}$.

5. Sampling formulas for the space B_ω

In this section we shall apply the previous results to oversampling formulas for the Hilbert transform sampling and the derivative sampling in B_ω. In the derivative sampling formula the coefficients are the values of the function and of its derivatives $f^{(j)}$, $1 \le j \le K$, at the points of a uniform grid on $\mathbb{R}$. It was first obtained by D. Jagerman and L. Fogel for $K = 1$ and by Linden and N. M. Abramson for any K [JF] [L] [LA]. Successively J. R. Higgins derived the same expansion formulas by using the Riesz basis method [Hi].
In [SF] D.M.S. Santos and P.J.S.G. Ferreira have obtained a two-channel derivative oversampling formula for B_{ω_a} with $\omega_a < \omega$ by projecting both the Riesz basis generators of the space B_ω and their duals into the space B_{ω_a}. With this technique the projected family is a frame; however notice that projecting the dual of a Riesz basis does not yield the dual frame. Thus the coefficients of the expansions of a function computed with respect to the projected duals are not minimal in least square norm.

Let t_o be such that $\omega \le h < 2\omega$ and let $\Phi = (\varphi_1, \varphi_2)$ be a vector such that E_{Φ,t_o} is a frame for B_ω. Then by (2.5)

$$f = \sum_{i=1,2}\sum_{k\in\mathbb{Z}} \langle f, \tau_{kt_o}\varphi_i\rangle \tau_{kt_o}\varphi_i^* \qquad \forall f \in B_\omega. \tag{5.1}$$

By using the Plancherel and the inversion formulas we see that the coefficients

$$\langle f, \tau_{kt_o}\varphi_i\rangle = \big(\mathcal{M}_j f\big)(kt_o) \qquad j = 1, 2 \tag{5.2}$$

are the samples of the functions $\mathcal{M}_j f = \mathcal{F}^{-1}\widehat{\varphi}_j\, \mathcal{F} f$ at the points kt_o, $k \in \mathbb{Z}$. For this reason (5.1) is called a *sampling formula.* These formulas are useful in applications when one wants to reconstruct a signal from samples taken from two transformed version of the signal. For instance one may want to reconstruct f from samples of f and f' (derivative sampling) or from samples of f and its Hilbert transform $\mathcal{H}f = -i\,\mathcal{F}^{-1}\text{sign}\,\mathcal{F}f$ (Hilbert transform sampling). Both are particular cases of the family of frames generated by the translates of two functions φ_1, φ_2 such that $\widehat{\varphi_1} = \chi_{[-\omega,\omega]}$, $\widehat{\varphi_2} = m\chi_{[-\omega,\omega]}$, where m is a function in $L^\infty(\mathbb{R})$.

Proposition 5.1. *Let m be a function in $L^\infty(\mathbb{R})$ and let $\Phi = (\varphi_1, \varphi_2)$ where*

$$\widehat{\varphi_1} = \chi_{[-\omega,\omega]} \qquad \widehat{\varphi_2}(x) = m\chi_{[-\omega,\omega]}. \tag{5.3}$$

Suppose that $\omega \le h < 2\omega$. Then E_{Φ,t_o} is a frame for B_ω if and only if there exists a positive number η such that

$$|m - \tau_{-h}m| \ge \eta \qquad a.e.\ in\ (h - \omega, \omega). \tag{5.4}$$

The Fourier transforms of the dual generators are

$$\widehat{\varphi_1}^* = \begin{cases} \frac{1}{h}(\tau_h \overline{m})(\tau_h \overline{m} - \overline{m})^{-1} & in \ \ [-\omega, \omega - h] \\ \\ \frac{1}{h}(1+|m|^2)^{-1} & in \ \ (\omega - h, h - \omega) \\ \\ -\frac{1}{h}(\tau_{-h} \overline{m})(\overline{m} - \tau_{-h} \overline{m})^{-1} & in \ \ [h-\omega, \omega] \end{cases}$$

$$\widehat{\varphi_2}^* = \begin{cases} -\frac{1}{h}(\tau_h \overline{m} - \overline{m})^{-1} & in \ \ [-\omega, \omega - h] \\ \\ \frac{1}{h} m(1+|m|^2)^{-1} & in \ \ (\omega - h, h - \omega) \\ \\ \frac{1}{h}(\overline{m} - \tau_{-h} \overline{m})^{-1} & in \ \ [h-\omega, \omega]. \end{cases}$$

If $h = \omega$ then E_{Φ, t_o} is a Riesz basis for B_ω.

Proof. Since $\det J^*_{\Phi,t_o} = h(\overline{m} - \tau_{-h}\overline{m})$ the assumptions of Theorem 3.6 are satisfied. The expression of the Fourier transforms of the dual generators can be easily obtained from Theorem 4.1. □

By choosing $m(x) = -i\,\mathrm{sign}(x)$ in (5.3) we obtain the Hilbert transform frames for B_ω. For $h = \omega$ the associated sampling formula is known as the *Hilbert transform sampling formula* (see [Hi, Ex.12.9]). The coefficients of the expansion are the values of the function f and its Hilbert transform $\mathcal{H}f$ at the sample points $kt_o, k \in \mathbb{Z}$. Denote by $\mathrm{sinc}(x)$ the function $\sin(x)/x$.

Corollary 5.2. *Let φ_1, φ_2 be defined by*

$$\widehat{\varphi_1} = \chi_{[-\omega,\omega]} \qquad \widehat{\varphi_2} = -i\chi_{[-\omega,\omega]}\,\mathrm{sign}. \tag{5.5}$$

If $\omega \le h < 2\omega$ then E_{Φ,t_o} is a tight frame for B_ω. The dual generators are $\varphi_i^* = (2h)^{-1}\varphi_i$ for $i = 1, 2$. If $h = \omega$ then E_{Φ,t_o} is a Riesz basis for B_ω. Moreover for any $f \in B_\omega$ the following Hilbert transform sampling formula holds

$$f(x) = \frac{\omega}{h}\sum_{k\in\mathbb{Z}}\Big(f(kt_o)\tau_{-kt_o}\cos\big(\frac{\omega x}{2}\big)\,\mathrm{sinc}\big(\frac{\omega x}{2}\big) - (\mathcal{H}f)(kt_o)\tau_{-kt_o}\sin\big(\frac{\omega x}{2}\big)\,\mathrm{sinc}\big(\frac{\omega x}{2}\big)\Big).$$

Proof. The assumptions of Proposition 5.1 are satisfied. Thus E_{Φ,t_o} is a frame and the expression of the duals follows immediately. The frame is tight because $TT^* = (2h)^{-1}I$, since $\Phi^* = (2h)^{-1}\Phi$.
Standard calculations show that if φ_1 and φ_2 are the functions given by (5.5) then

$$\varphi_1(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\omega\cos\big(\frac{\omega x}{2}\big)\,\mathrm{sinc}\big(\frac{\omega x}{2}\big) \qquad \varphi_2(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\omega\sin\big(\frac{\omega x}{2}\big)\,\mathrm{sinc}\big(\frac{\omega x}{2}\big). \tag{5.6}$$

Moreover the coefficients of the expansion formula (5.1) are $\sqrt{2\pi}\hat{f}(kt_o)$ and $-\sqrt{2\pi}(\mathcal{H}f)(kt_o)$, $k \in \mathbb{Z}$. This proves also the expansion formula. □

By choosing $m(x) = ix$ in (5.5) we obtain the derivative frame for B_ω. Given a function g we shall denote by g_δ the function $g(\delta x)$. Note that $\delta\,\mathrm{sinc}_\delta = \sqrt{\pi/2}\widehat{\chi}_{[-\delta,\delta]}$.

Corollary 5.3. *Let φ_1, φ_2 be defined by*

$$\widehat{\varphi_1} = \chi_{[-\omega,\omega]} \qquad \widehat{\varphi_2} = ix\chi_{[-\omega,\omega]}. \tag{5.7}$$

If $\omega \leq h < 2\omega$ then E_{Φ,t_o} is a frame for B_ω; if $h = \omega$ then it is a Riesz basis for B_ω. The Fourier transforms of the dual generators are

$$\widehat{\varphi_1}^*(x) = \begin{cases} \frac{1}{h}(1 - \frac{|x|}{h}) & \text{if } \; h - \omega < |x| < \omega \\ \\ \frac{1}{h}(1 + x^2)^{-1} & \text{if } \; |x| < h - \omega \end{cases}$$

$$\widehat{\varphi_2}^*(x) = \begin{cases} \frac{i}{h^2}\,\mathrm{sign}(x) & \text{if } \; h - \omega < |x| < \omega \\ \\ i\frac{x}{h}(1 + x^2)^{-1} & \text{if } \; |x| < h - \omega. \end{cases}$$

Moreover for any $f \in B_\omega$ the following derivative sampling formula holds

$$f = \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \Big(f(kt_o)\tau_{-kt_o}\varphi_1^* - f'(kt_o)\tau_{-kt_o}\varphi_2^* \Big), \tag{5.8}$$

where

$$\varphi_1^*(x) = e^{-|\cdot|} * (h-\omega)\frac{1}{h}\mathrm{sinc}_{h-\omega}(x) + \frac{1}{\pi}\big(\omega\,\mathrm{sinc}_\omega - (h-\omega)\,\mathrm{sinc}_{h-\omega}\big) * \mathrm{sinc}^2_{h/2}(x)$$

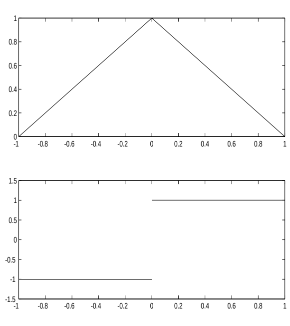
$$\varphi_2^*(x) = e^{-|\cdot|} * (h-\omega)\frac{1}{h}\mathrm{sinc}'_{h-\omega}(x) + \frac{\sqrt{2}}{\sqrt{\pi}}\frac{1}{xh^2}\big(\cos_\omega - \cos_{h-\omega}\big)(x).$$

Proof. By Proposition 5.1 E_{Φ,t_o} is a frame for all $\omega \leq h < 2\omega$ and it is a Riesz basis if $h = \omega$. From (5.2) we see that the coefficients of the expansion (5.1) are $\sqrt{2\pi}\hat{f}(kt_o)$ and $-\sqrt{2\pi}f'(kt_o)$. The expression of the dual generators can be obtained from $\widehat{\varphi_1}^*$ and $\widehat{\varphi_2}^*$ by computing the inverse Fourier transform. $\square$

A simple calculation shows that if $h = \omega$ then

$$\varphi_1^*(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{sinc}^2(\frac{\omega x}{2}) \qquad \varphi_2^*(x) = -\frac{1}{\sqrt{2\pi}}\,x\,\mathrm{sinc}^2(\frac{\omega x}{2}).$$

Figure 1 and Figure 2 below show the Fourier transforms of the dual generators in the Riesz basis case $h = \omega = 1$, and in the case $\omega = 1$ and $h = 3\omega/2 = 3/2$ respectively.



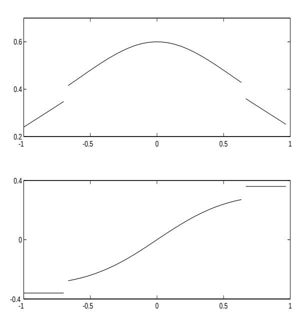


Figure 1: $\widehat{\varphi_1^\star}, \frac{1}{i}\widehat{\varphi_2^\star}$ for $h = \omega = 1$ Figure 2: $\widehat{\varphi_1^\star}, \frac{1}{i}\widehat{\varphi_2^\star}$ for $h = \frac{3}{2}\omega = \frac{3}{2}$

Remark In [JF] Jagerman and Fogel proved the following two-channel derivative sampling formula in the case $h=\omega$ i.e. $t_o = 2\pi/\omega$

$$f(x) = \sum_k f(kt_o)\Big(\operatorname{sinc}^2\frac{\omega}{2}\big(x-kt_o\big) \;+\; \frac{2}{\omega}f'(kt_o)\sin\frac{\omega}{2}\big(x-kt_o\big)\ \operatorname{sinc}\frac{\omega}{2}\big(x-kt_o\big)\Big)$$

(see also [Hi, p.135]). Thus formula (5.8) is an extension of the case $t_o = 2\pi/\omega$ to all values of $t_o \in [\pi/\omega, 2\pi/\omega)$ (i.e. for all h such that $\omega \le h < 2\omega$).

Our last example is a three channel derivative oversampling formula. To obtain a frame with three generators we must choose $\frac{2}{3}\omega \le h < \omega$ i.e. $t_o \in \big[3\pi/\omega, 2\pi/\omega\big)$.

Corollary 5.4. *Let $\widehat{\varphi_1}, \widehat{\varphi_2}, \widehat{\varphi_3}$ be defined by*

$$\widehat{\varphi_1} = \chi_{[-\omega,\omega]}, \qquad \widehat{\varphi_2} = ix\chi_{[-\omega,\omega]}, \qquad \widehat{\varphi_3} = -x^2\chi_{[-\omega,\omega]}.$$

If $\frac{2}{3}\omega \le h < \omega$ then E_{Φ,t_o} is a frame for B_ω; if $h = \frac{2}{3}\omega$ then it is a Riesz basis for B_ω. The Fourier transform of the dual generators are

(5.9)

$$\widehat{\Phi^*}(x) = \begin{cases} \frac{1}{2}h^{-3}\big(x^2+3hx+2h^2, -i(2x+3h), -1\big) & \textit{if} \quad -\omega < x < \omega - 2h \\ \big(A_h(x), B_h(x), C_h(x)\big) & \textit{if} \quad \omega - 2h < x < h - \omega \\ h^{-3}\big(h^2 - x^2, 2ix, 1\big) & \textit{if} \quad h-\omega < x < \omega - h \\ \big(A_{-h}(x), B_{-h}(x), C_{-h}(x)\big) & \textit{if} \quad \omega - h < x < 2h - \omega \\ \frac{1}{2}h^{-3}\big(x^2-3hx+2h^2, -i(2x-3h), -1\big) & \textit{if} \quad 2h-\omega < x < \omega \end{cases}$$

where

$$A_h(x) = \frac{1}{h^2}\,\frac{(x+h)+(2x+h)(x+h)^2}{1+(2x+h)^2+x^2(x+h)^2} \qquad B_h(x) = \frac{-i}{h^2}\,\frac{1-x(x+h)^3}{1+(2x+h)^2+x^2(x+h)^2}$$

$$C_h(x) = \frac{1}{h^2}\,\frac{h+2x+x(x+h)^2}{1+(2x+h)^2+x^2(x+h)^2}.$$

Note that if $h = \frac{2}{3}\omega$ the intervals $(\omega - h, 2h-\omega)$ and $(2h-\omega, h-\omega)$ are empty.

Proof. The family E_{Φ,t_o} is a frame for B_ω by Theorem 4.4. To compute the dual generators we used formula (4.12). ☐

Thus for each value of the parameter h in $[\frac{2}{3}\omega, \omega)$ the family E_{Φ,t_o} is a frame and every signal in B_ω can be reconstructed from the values $f(kt_o), f^{(1)}(kt_o), f^{(2)}(kt_o)$, $k \in \mathbb{Z}$, by the following three-channel derivative sampling formula

$$f = \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{3}\sum_{k\in\mathbb{Z}}(-1)^{i-1}f^{(i-1)}(kt_o)\varphi_i^*(x-kt_o), \tag{5.10}$$

where the dual generators are given by (5.9).

In the rest of this section we present two numerical experiments of reconstruction of a band limited signal by using formula (5.10). We have chosen the signal $f(x) = (2\pi)^{-1/2}(\operatorname{sinc}(x/2))^2$ (see Figure 3); note that the function f is the Fourier transform of the function $(1-|x|)_+$, therefore $\omega = 1$. In the first experiment we take $h = 2\omega/3$ so that E_{Φ,t_o} is a Riesz basis; in the second experiment we take $h = 11\omega/15$ so that E_{Φ,t_o} is a frame. We observe that in the first case it is possible

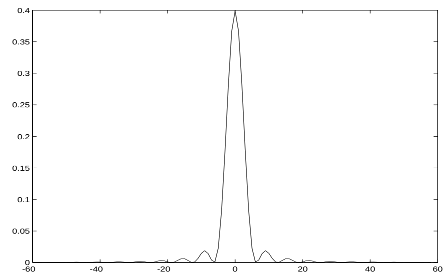


Figure 3: Signal $f(x) = \frac{1}{\sqrt{2\pi}}(\operatorname{sinc}(x/2))^2$

to find the analytic expression of the functions φ_j^* in (5.9), while in the second case they must be computed numerically. Indeed for $h = 2\omega/3$ one obtains

$$\begin{aligned}
\varphi_1^* &= \frac{1}{\sqrt{2\pi}}\Big(1 + \frac{\omega^2 x^2}{18}\Big)\operatorname{sinc}^3\Big(\frac{\omega}{3}x\Big) \\
\varphi_2^* &= -\frac{1}{\sqrt{2\pi}} x \operatorname{sinc}^3\Big(\frac{\omega}{3}x\Big) \\
\varphi_3^* &= \frac{1}{2\sqrt{2\pi}} x^2 \operatorname{sinc}^3\Big(\frac{\omega}{3}x\Big).
\end{aligned}$$

Figures 4 and 5 below show the duals $\varphi_1^*, \varphi_2^*, \varphi_3^*$ and $\widehat{\varphi_1^*}$, $\frac{1}{i}\widehat{\varphi_2^*}, \widehat{\varphi_3^*}$ for $h = \frac{2}{3}\omega$, $\omega = 1$. Since for $h = \frac{2}{3}\omega$ the duals are known, one can write explicitly the sampling formula

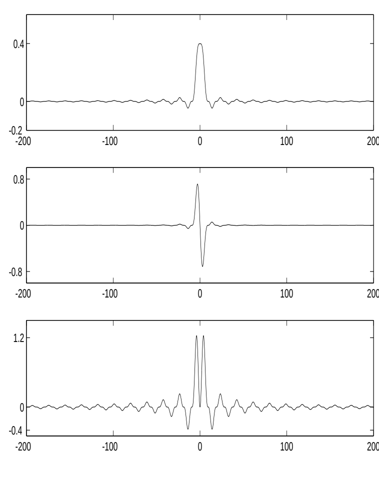


Figure 4: $\Phi^\star$; $h = \frac{2}{3}\omega, \omega = 1$

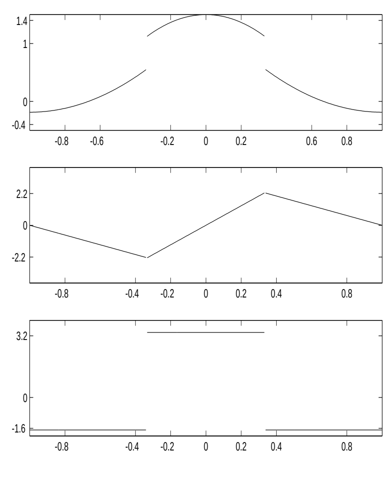


Figure 5: $\widehat{\varphi_1^*}$, $\frac{1}{i}\widehat{\varphi_2^*}$, $\widehat{\varphi_3^*}$; $h = \frac{2}{3}\omega$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \sin^3\Big(\frac{\omega x}{3}\Big) \sum_n \big(-1\big)^n \Big[f\Big(\frac{3n\pi}{\omega}\Big) \frac{1}{\big(x - \frac{3n\pi}{\omega}\big)^3} +$$

$$+ f\Big(\frac{3n\pi}{\omega}\Big) \frac{\omega^2/9}{\big(x - \frac{3n\pi}{\omega}\big)} - f^{(1)}\Big(\frac{3n\pi}{\omega}\Big) \frac{1}{\big(x - \frac{3n\pi}{\omega}\big)^2} + f^{(2)}\Big(\frac{3n\pi}{\omega}\Big) \frac{1}{2\big(x - \frac{3n\pi}{\omega}\big)^2} \Big].$$

This formula was first given by Linden in [L] and Linden and Abramson in [LA], see also [HS].

In the case $h = \frac{11}{12}\omega$. $\omega = 1$, the dual generators have been obtained by computing numerically the inverse Fourier transforms of the functions in (5.9): Figure 6 and Figure 7 below show the functions $\varphi_1^*, \varphi_2^*, \varphi_3^*$ and $\widehat{\varphi_1^*}$, $\frac{1}{i}\widehat{\varphi_2^*}, \widehat{\varphi_3^*}$; $h = \frac{11}{15}\omega,\ \ \omega = 1$.

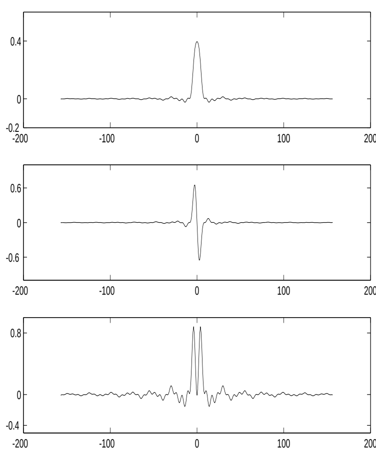


Figure 6: $\Phi^\star$, $h = \frac{11}{15}\omega$, $\omega = 1$

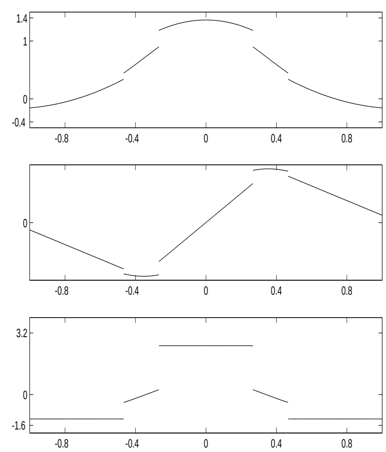


Figure 7: $\widehat{\varphi_1^*}$, $\frac{1}{i}\widehat{\varphi_2^*}, \widehat{\varphi_3^*}$; $h = \frac{11}{15}\omega$

Figure 8 and Figure 9 show the error in the cases $h = 2\omega/3$ and $h = 11\omega/15$, respectively, $\omega = 1$. Notice that in the first case the order of magnitude of the error is 10^{-4} while in the second case is 10^{-3}. In second case, since the functions φ_j^* were computed numerically, to compute their values at the points $x - kt_o$ we used spline interpolation. This accounts for the different order of magnitude of the error.

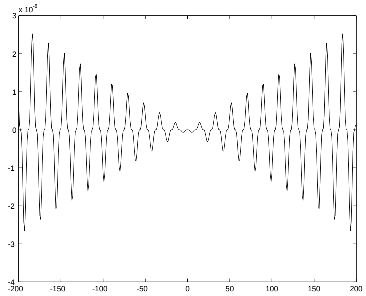


Figure 8: The error with $h = \frac{2}{3}$.

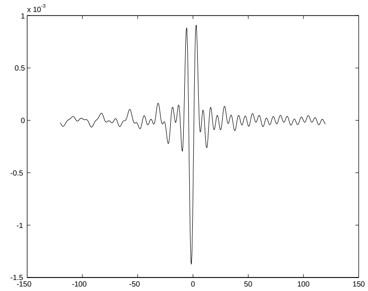


Figure 9: The error with $h = \frac{11}{15}$.

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